LIMITS OF HODGE STRUCTURES VIA HOLONOMIC D-MODULES

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Abstract. We construct the limiting mixed Hodge structure of a degeneration of compact Kähler manifolds over the unit disk with a possibly non-reduced normal crossing singular central fiber via holonomic $\mathscr{D}$-modules, which generalizes some results of Steenbrink. Our limiting mixed Hodge structure does not carry a $\mathbb{Q}$-structure; instead we use sesquilinear pairings on $\mathscr{D}$-modules as a replacement. The associated graded quotient of the weight filtration of the limiting mixed Hodge structure can be computed by the cohomology of the cyclic coverings of certain intersections of components of the central fiber.

1. Introduction

1.1. **Limits of Hodge structures.** Consider a degeneration of compact Kähler manifolds $f : X \to \Delta$ over the unit disk Δ. The cohomology of each smooth fiber carries a polarizable Hodge structure. It is natural to ask how the family of Hodge structures on the cohomologies of smooth fibers degenerate and how the cohomology of the central fiber relates to that of nearby fibers. These are two classical and central questions in Hodge theory. Before Saito's theory of mixed Hodge modules [Sai88, Sai90], Schmid showed the existence of a limiting mixed Hodge structure for an abstract polarized variation of Hodge structures over the unit disk [Sch73] using Lie theoretic methods, and later Cattani, Kapplan and Schmid extend this to polydisks [CKS86]. For the variation of Hodge structures coming from a semistable family of Kähler manifolds over a 1-dimensional base, the limiting mixed Hodge structure was first established by Steenbrink [Ste76] whose construction is equivalent to Schmid's in [Sch73] but purely geometric:

Theorem (Steenbrink). *Let $f : X \to \Delta$ be a proper holomorphic morphism which is smooth away from the origin, whose central fiber Y is reduced simple normal crossing. Suppose X is Kähler of dimension $n+1$. Then the hypercohomology $H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ of the relative log de Rham complex restricted on Y admits a limiting mixed Hodge structure with a $\mathbb{Q}$-structure whose graded quotient of the weight filtration can be expressed in terms of the cohomology of certain intersections of components of Y via spectral sequences.*

Let us briefly explain Steenbrink's result. Suppose we are in the setting of the theorem but Y is possibly non-reduced. Denote by $X^* = X \smallsetminus Y$ and $\Delta^* = \Delta \smallsetminus \{0\}$. Then the higher direct image of the relative de Rham complex $R^k f_* \Omega^{\bullet+n}_{X^*/\Delta^*}$ is a vector bundle, where the shifting is needed to adopt the convention of the theory of perverse sheaves and $\mathscr{D}$-modules; it underlies a polarizable variation of Hodge structure of weight n over the punctured disk Δ^*. Recall that a *polarized variation of Hodge structure* of weight n over a complex manifold Z is an integrable connection $(\mathcal{V}, \nabla)$ together with a so-called Hodge filtration by subundles $F^\bullet\mathcal{V}$ and a flat Hermitian pairing $S : \mathcal{V} \otimes_{\mathbb{C}} \overline{\mathcal{V}} \to \mathscr{C}^\infty_Z$ satisfying (1) Griffith transversality $\nabla F^\bullet \mathcal{V} = \Omega_Z \otimes F^{\bullet-1}\mathcal{V}$, and (2) each fiber of $(\mathcal{V}, F^\bullet\mathcal{V}, S)$ is a polarized Hodge structure of weight n. However, the higher direct image of the relative de Rham complex $\Omega^{\bullet+n}_{X/\Delta}$ does not give anything interesting when Y is singular. Steenbrink discovered a natural extension of the vector bundle $R^k f_* \Omega^{\bullet+n}_{X^*/\Delta^*}$ over the origin via the relative log de Rham complex. Let

$$\Omega_{X/\Delta}(\log Y) = \Omega_X(\log Y)/f^*\Omega_\Delta(\log 0) \text{ and } \Omega^p_{X/\Delta}(\log Y) = \bigwedge^p \Omega_{X/\Delta}(\log Y),$$

Date: September 1, 2021.

where $\Omega_X(\log Y)$ is the sheaf of meromorphic one-forms with log poles along Y. Then the *relative log de Rham complex* is defined to be

$$\Omega^{\bullet+n}_{X/\Delta}(\log Y) = \{\mathscr{O}_X \to \Omega_{X/\Delta}(\log Y) \to \cdots \to \Omega^n_{X/\Delta}(\log Y)\}[n].$$

Steenbrink showed in [Ste76] that $R^k f_*\Omega^\bullet_{X/\Delta}(\log Y)$ is a locally free integrable logarithmic connection with a pole along the origin whose residue R has eigenvalues in $[0,1)\cap\mathbb{Q}$ for each $k\in\mathbb{Z}$. It follows from Grauert's theorem that there exists a canonical isomorphism

$$R^k f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)\otimes\mathbb{C}(p)\simeq H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_{X_p})$$

for every fiber X_p over any point $p\in\Delta$, where $\mathbb{C}(p)$ denotes the residue field of p. The vector bundle $R^k f_*\Omega^\bullet_{X/\Delta}(\log Y)$ is Deligne's canonical extension [Del70] of $R^k f_*\Omega^{\bullet+n}_{X^*/\Delta^*}$ with eigenvalues of the residues of the log connection in the interval in $[0,1)$. Now we can think of the space $H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ as a canonical specialization of $H^k(X_p,\Omega^{\bullet+n}_{X_p})$ for general fibers X_p. In fact, the limiting Hodge filtration is induced by the stupid filtration defined by,

$$F^{-\ell}\Omega^{\bullet+n}_{X/\Delta}(\log Y) = \{\Omega^{-\ell}_{X/\Delta}(\log Y)\to\cdots\to\Omega^n_{X/\Delta}(\log Y)\}[n+\ell],$$

for each $\ell\in\mathbb{Z}$. This extends the Hodge filtration $F^\bullet R^k f_*\Omega^{\bullet+n}_{X^*/\Delta^*}$ for the variation of Hodge structure $R^k f_*\Omega^{\bullet+n}_{X^*/\Delta^*}$ which is also induced by the stupid filtration on the complex $\Omega^{\bullet+n}_{X^*/\Delta^*}$. When Y is reduced, the residue R is nilpotent on the hypercohomology of $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ for every k so it gives a monodromy filtration $W_\bullet = W_\bullet(R)$ on $H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ uniquely characterized by two properies: (1) $RW_\bullet\subset W_{\bullet-2}$ and (2) $R^r:\mathrm{gr}^W_r\to\mathrm{gr}^W_{-r}$ is an isomorphism for every $r\geq 0$. The filtration $W_\bullet(R)$ is called the monodromy filtration because $\exp\left(-2\pi\sqrt{-1}R\right)$ is the monodromy induced by the generator of the fundamental group of Δ^*. Steenbrink showed that the monodromy filtration is the weight filtration of the limiting mixed Hodge structure when f is projective, and this was later generalized to the Kähler case by Guillén and Navarro Aznar in [GNA90].

Steenbrink later pointed out the limiting mixed Hodge structure he constructed only depends on the log structure associated to the semistable family $f:X\to\Delta$ [Ste95]. Inspired by the idea in [Ste95], Fujisawa extended Steenbrink's results in [Ste76, Ste95] to semistable Kähler families over the polydisk and furthermore to the log geometry setting [Fuj99, Fuj08, Fuj14]. Recently, Nakkajima announced a simpler proof of Fujisawa's results [Nak21].

1.2. **Main results.** We revisit Steenbrink's theorem and construct the limiting mixed Hodge structure of the degeneration over the unit disk Δ with a simple normal crossing central fiber Y which is possibly non-reduced via the theory of holonomic $\mathscr{D}$-modules. Although we can run Mumford's semistable reduction [KKMSD73], which is a sequence of base changes, normalizations and blow-ups, on every degeneration of Kähler manifolds over the unit disk to obtain a semistable degeneration, it is still interesting to remove the assumption that Y is reduced in Steenbrink's theorem since the semistable reduction may not be canonical. When Y is non-reduced, the residue is no longer nilpotent; instead, we need to consider the Jordan-Chevallay decomposition of R. Here is our main theorem:

Theorem A. *Let $f:X\to\Delta$ be a proper holomorphic morphism which smooth away from the origin, whose central fiber Y is possibly non-reduced simple normal crossing. Assume that X is Kähler of dimension $n+1$. Let R_n (resp. R_s) denote the nilpotent (resp. semisimple) part of the Jordan-Chevalley decomposition of the residue operator R on $\bigoplus_k H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$. Then each eigenspace of R_s on*

$$\bigoplus_{k,\ell}\mathrm{gr}^W_\ell H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$$

underlies a limiting polarized bigraded Hodge-Lefschetz structure over $\mathbb{C}$ of central weight n, where $W_\bullet = W_\bullet(R_n)$ is the monodromy filtration associated to R_n.

A polarized bigraded Hodge-Lefschetz structure is essentially a direct sum of polarized Hodge structures of different weights preserving by an $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-action. In the setting of Theorem A, the $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-action will be induced by the operator R_n and $2\pi\sqrt{-1}L$ where $L = \omega\wedge$ is the Lefschetz operator for a Kähler form ω. In particular, each component $\mathrm{gr}^W_\ell H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ is a Hodge structure of weight $n+k+\ell$ and there are two Hard Lefschetz type isomorphisms of Hodge structures:

- $\left(2\pi\sqrt{-1}L\right)^k : \mathrm{gr}^W_\ell H^{-k}(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \to \mathrm{gr}^W_\ell H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)(k)$ for $k \geq 0, \ell \in \mathbb{Z}$;
- $R_n^\ell : \mathrm{gr}^W_\ell H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \to \mathrm{gr}^W_{-\ell} H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)(-\ell)$ for $\ell \geq 0, k \in \mathbb{Z}$.

Theorem A implies that each $H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ still underlies a limiting mixed Hodge structure of weight $n+k$ whose weight filtration is given by $W_\bullet = W_\bullet(R_n)$ when the central fiber is non-reduced. We refer to §2.4 for the definition of polarized bigraded Hodge-Lefschetz structures. Our argument also says that the limiting mixed Hodge structure can be computed in terms of the cohomology of certain cyclic coverings of intersections of components of Y via spectral sequences.

Steenbrink proved, loosely speaking, that $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ is isomorphic to $\psi_f(\mathbb{C}_X[n+1])$ in the derived category of complex vector spaces $\mathbf{D}^b(X,\mathbb{C})$ where ψ_f denotes the nearby cycles functor, so that the function $p \mapsto \dim H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_{X_p})$ is constant on Δ. Thanks to Grauert's theorem, the sheaf $R^k f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is locally free. The log connection on $R^k f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is the higher direct image of an operator $\nabla \in \mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$, which fits in a distinguished triangle in $\mathbf{D}^b(X,\mathbb{C})$

$$f^*\Omega_\Delta \otimes \Omega^{\bullet+n-1}_{X/\Delta}(\log Y) \longrightarrow \Omega^{\bullet+n}_X(\log Y) \longrightarrow \Omega^{\bullet+n}_{X/\Delta}(\log Y) \xrightarrow{\nabla} f^*\Omega_\Delta \otimes \Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

if we trivialize $f^*\Omega_\Delta$. The induced operator $[\nabla] : \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y \to \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ has a characteristic polynomial whose roots are in $[0,1)\cap\mathbb{Q}$. The action of $[\nabla]$ on the hypercohomology of $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ is identical to the residue operator R of the log connection. So the methods of studying the monodromy filtration of R on the cohomology is to make the monodromy filtration of $[\nabla]$ on the complex $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ explicit. One of the main difficulties that we encounter is the construction of the rational monodromy filtration on the complex $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ because the operator $[\nabla]$ only lives in the derived category. Steenbrink resolves the relative log de Rham complex using a certain double complex and then he works out the monodromy filtration directly in the case that Y is reduced. He also needs to show that the monodromy filtrations are defined over $\mathbb{Q}$, using some complicated topological argument, so that all the data gives a rational cohomological mixed Hodge complex.

Through the Riemann-Hilbert correspondence [Kas84, Meb84], there should be a regular holonomic $\mathscr{D}$-module whose de Rham complex is isomorphic to $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ in $\mathbf{D}^b(X,\mathbb{C})$ since the nearby cycle functor preserves perversity [Beĭ87]. On the $\mathscr{D}$-module side, we can derive the monodromy filtration easily by local calculations on a single $\mathscr{D}$-module which bypasses the derived categories. More importantly, we give a concrete description of the primitive parts of the associated quotient of the monodromy filtrations. Instead of using $\mathbb{Q}$-structures, we consider sesuqilinear pairings on $\mathscr{D}$-modules, which play the role of a polarization on a Hodge structure. In fact, the polarization on the bigraded Hodge-Lefschetz structure in Theorem A will be induced by a sesquilinear pairing. Although part of the topological data is lost, the sesquilinear pairings that we shall use can be constructed pure algebraically and only involve symbolic calculations. The local calculation and the sesquilinear pairing justify the fact that the monodromy filtration of $[\nabla]$ is the correct choice for the weight filtration. Our method also allows us to construct naturally the limit when Y is non-reduced.

As an application, we establish the local invariant theorem, which is a piece in the Clemens-Schmid sequence [Cle77], when Y is non-reduced. The local invariant cycle theorem first was proved by Deligne in an algebraic setting when the base is a scheme [Del71, Theorem 4.1.1] and later treated in [Ste76], [Cle77] and [GNA90] for a semistable Kähler degeneration. It also generalized to mixed Hodge module theory by Saito [Sai88, Sai90].

Theorem B (local invariant cycle theorem)**.** *Suppose we are in the same setting as in Theorem A. Then the following sequence of mixed Hodge structures is exact:*

$$H^\ell(Y,\mathbb{C}) \longrightarrow H^\ell(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \xrightarrow{R} H^\ell(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)(-1).$$

In other words, all cohomology classes invariant under the monodromy action comes from the cohomologies of Y.

1.3. **Strategy of the construction.** Let $f : X \to \Delta$ be a proper holomorphic morphism smooth away from the origin such that the central fiber Y is simple normal crossing but not necessarily reduced. Assume that X is Kähler of dimension $n+1$ and $Y = \sum_{i\in I} e_i Y_i$ where the Y_i's are smooth components and I a finite index set. We adopt the convention that if $F^\bullet$ denotes a decreasing filtration then $F_{-\bullet} = F^\bullet$ denotes the corresponding increasing filtration and vice versa.

We first give a different proof of the local freeness of $R^k f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ which only uses the fact that $[\nabla]$ has eigenvalues in $[0,1)$ (Theorem 3.2). Then we translate the data of the relative log de Rham complex to the $\mathscr{D}$-module side (see §4):

Theorem C. *There exists a filtered holonomic $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ whose de Rham complex $\mathrm{DR}_X\mathcal{M}$ with the induced filtration $F_\bullet \mathrm{DR}_X\mathcal{M}$ is isomorphic to $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ with the stupid filtration in the derived category of filtered complex of $\mathbb{C}$-vector spaces. Moreover, there exists an operator $R\colon (\mathcal{M}, F_\bullet\mathcal{M}) \to (\mathcal{M}, F_{\bullet+1}\mathcal{M})$ whose eigenvalues are in $[0,1)\cap\mathbb{Q}$ such that $\mathrm{DR}_X R$ can be identified with $[\nabla]$ via the above isomorphism.*

Then we investigate the Jordan block of the operator R. Let $\mathcal{M}_{\geq\alpha}$ (resp. $\mathcal{M}_{>\alpha}$) be the submodule of $\mathcal{M}$ spaned by the generalized eigen-modules $\ker(R-\lambda)^\infty$ for $\lambda \geq \alpha$ (resp. $\lambda > \alpha$). Let $\mathcal{M}_\alpha = \mathcal{M}_{\geq\alpha}/\mathcal{M}_{>\alpha}$. Note that $\mathcal{M}_\alpha$ is canonically isomorphic to $\ker(R-\alpha)^\infty$ and therefore $R_\alpha = R-\alpha$ acts nilpotently on $\mathcal{M}_\alpha$. Using an idea of Saito [Sai90], we filter $\mathcal{M}_\alpha$ by

$$F_\ell\mathcal{M}_\alpha = \frac{F_\ell\mathcal{M}\cap\mathcal{M}_{\geq\alpha}+\mathcal{M}_{>\alpha}}{\mathcal{M}_{>\alpha}}, \qquad \text{for } \ell\in\mathbb{Z}.$$

The filtration $F_\bullet\mathcal{M}_\alpha$ is different from the naive one $F_\bullet\mathcal{M}\cap\ker(R-\alpha)^\infty$. The reason why we do not use the naive filtration is that $F_\bullet\mathcal{M}_\alpha$ not only gives the correct weight but is also easy to work out. We prove that any power of the operator R_α is strict with respect to $F_\bullet\mathcal{M}_\alpha$. Namely, for every $\ell\geq 0$, we have the relation $R^\ell_\alpha F_\bullet\mathcal{M}_\alpha = F_{\bullet+\ell}\mathcal{M}\cap R^\ell_\alpha\mathcal{M}_\alpha$ (Theorem 5.1 for the case Y is reduced and Theorem 7.5 for the general case). This implies that the monodromy filtration $W_\bullet\mathcal{M}_\alpha$ and $F_\bullet\mathcal{M}_\alpha$ interacts very well. Note that the monodromy filtration associated to R_α is the same as the one of R_n on $\mathcal{M}_\alpha$, the nilpotent part of R in Jordan-Chevalley decomposition. We have the induced good filtrations

$$F_\bullet W_r\mathcal{M}_\alpha = F_\bullet\mathcal{M}\cap W_r\mathcal{M}_\alpha \qquad \text{and} \qquad F_\bullet \mathrm{gr}^W_r\mathcal{M}_\alpha = F_\bullet W_r\mathcal{M}_\alpha / F_\bullet W_{r-1}\mathcal{M}_\alpha.$$

Denote by $\mathcal{P}_{\alpha,\ell} = \ker R^{\ell+1}_\alpha \cap \mathrm{gr}^W_\ell \mathcal{M}_\alpha$ the ℓ-th primitive for $\ell\geq 0$, which is isomorphic to

$$\frac{\ker R^{\ell+1}_\alpha}{\ker R^\ell_\alpha + \mathrm{im}\, R_\alpha \cap \ker R^{\ell+1}_\alpha}.$$

We endow it with the induced good filtration $F_\bullet\mathcal{P}_{\alpha,\ell} = \mathrm{im}\,(F_\bullet\mathcal{M}\cap\ker R^{\ell+1}_\alpha \to \mathcal{P}_{\alpha,\ell})$. As a corollary of the strictness of every power of R_α, the Lefschetz decomposition of $\mathrm{gr}^W\mathcal{M}_\alpha$ respects the good filtrations, i.e.

$$F_\bullet \mathrm{gr}^W_r\mathcal{M}_\alpha = \bigoplus_{\ell\geq 0,-\frac{r}{2}} R^\ell_\alpha F_{\bullet-\ell}\mathcal{P}_{\alpha,r+2\ell} \qquad \text{for } r\geq 0.$$

See Theorem 5.6 for the case Y is reduced and Theorem 7.8 for the general case. This corollary suggests that it suffices to study the hypercohomology of each primitive part. The primitive parts will be the source for the pure polarized Hodge structures.

We will construct a sesquilinear pairing $S_\alpha : \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}_\alpha} \to \mathfrak{C}_X$ using the Mellin transformation [Sab02], where $\overline{\mathcal{M}_\alpha}$ is the naive conjugation of $\mathcal{M}_\alpha$ and $\mathfrak{C}_X$ is the sheaf of currents. Both $\mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}_\alpha}$ and $\mathfrak{C}_X$ canonically carry $\mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_{\overline{X}}$-module structures where $\mathscr{D}_{\overline{X}}$ denotes the sheaf of anti-holomorphic differential operators and the sesquilinear pairing is just a morphism of $\mathscr{D}_X \otimes_{\mathbb{C}} \overline{\mathscr{D}_X}$-modules. A good reference is the MHM project [SS] by Sabbah and Schnell. The sesquilinear pairings on $\mathcal{M}_\alpha$ is an analogy of a polarization on a Hodge structure: a complex polarized Hodge structure of weight n can be described as a filtered vector space $(V, F^\bullet)$ with a Hermitian pairing S such that $(-1)^{n-p} S$ is a Hermitian inner product on $F^p \cap G^{n-p}$ where G^{n-p} is the S-orthogonal complement of F^{p+1}. The sesquilinear pairing S_α induces the second filtration on the hypercohomology of $\mathrm{DR}_X \mathcal{M}_\alpha$. For example, if Y is reduced, the pairing on $\mathcal{M}$ is induced by

$$\mathrm{Res}_{s=0} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \frac{d\bar{t}}{\bar{t}} \int_{X_t} : \Omega^n_{X/\Delta}(\log Y) \otimes_{\mathbb{C}} \overline{\Omega^n_{X/\Delta}(\log Y)} \to \mathfrak{C}_X,$$

where the constant scalar $\varepsilon(n+2)(2\pi\sqrt{-1})^{-(n+1)}$ depending on the dimension is used to make the pairing independent of the choice of orientation. The Mellin transformation is used here to extract the principal part of the asymptotic expansion of fiberwise integration $\int_{X_t} : \omega_{X_t} \otimes_{\mathbb{C}} \overline{\omega_{X_t}} \to \mathscr{C}_{X_t}$. We refer to the §2.1 for the definition of sesquilinear pairings on $\mathscr{D}$-module

The operator R_α is self-adjoint with respect to the pairing $S_\alpha : \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}}_\alpha \to \mathfrak{C}_X$, i,e, $S_\alpha(-, R_\alpha -) = S_\alpha(R_\alpha -, -)$. See §6 for the case that Y is reduced §8 for the general case. This implies we have an induced pairing on the associated graded modules:

$$S_{\alpha,r} : \mathrm{gr}^W_r \mathcal{M}_\alpha \otimes_{\mathbb{C}} \mathrm{gr}^W_{-r} \mathcal{M}_\alpha \to \mathfrak{C}_X.$$

Then $P_{R_\alpha} S_{\alpha,r} = S_{\alpha,r} \circ (\mathrm{id} \otimes_{\mathbb{C}} R^r_\alpha)$ defines a sesquilinear pairing on the primitive part $\mathcal{P}_{\alpha,r}$.

Theorem D. *The cohomologies of the de Rham complex of $\mathcal{P}_{\alpha,r}$*

$$\bigoplus_{\ell\in\mathbb{Z}} H^\ell(X, \mathrm{DR}_X \mathcal{P}_{\alpha,r})$$

together with the filtration induced by $F_\bullet \mathcal{P}_{\alpha,r}$ and the sesquilinear pairing induced by $P_{R_\alpha} S_{\alpha,r}$ determine a polarized Hodge-Lefschetz structure of central weight $n+r$ with $\mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$.

A polarized Hodge-Lefschetz structure basically is a direct sum of Hodge structures of different weights preserving by an $\mathfrak{sl}_2(\mathbb{C})$-action modeled by the direct sum of all the cohomologies of a compact Kähler manifold. We refer to §2.3 for the definition of polarized Hodge-Lefschetz structures. To illustrate the idea of Theorem D, assume for a moment that Y is reduced. Then the endomorphism R will be nilpotent and this implies that $\mathcal{M} = \mathcal{M}_0$. Denote by $Y^J = \bigcap_{i\in J} Y_i$ for any non-empty subset J of I. Let $\tau^J : Y^J \to X$ be the closed embedding and $\tau^{(r+1)} : \tilde{Y}^{(r+1)} = \coprod_{\# J = r+1} Y^J \to X$ be the natural morphism for every $r \geq 0$. For simplicity, suppose $\mathcal{P}_r = \mathcal{P}_{0,r}$. We will show that there exists a filtered isomorphism (Theorem 5.7)

$$\phi_r : (\mathcal{P}_r, F_\bullet \mathcal{P}_r) \to \tau^{(r+1)}_+ \omega_{\tilde{Y}^{(r+1)}}(-r).$$

Here, the Tate twist of a filtered $\mathscr{D}$-module is $(\mathcal{N}, F_\bullet \mathcal{N})(-r) = (\mathcal{N}, F_{\bullet+r}\mathcal{N})$. Moreover, the isomorphism respects the pairing $P_R S_r$ on $\mathcal{P}_r$ (Theorem 6.5):

$$P_R S_r(-,-) = \frac{(-1)^r}{(r+1)!} \tau^{(r+1)}_+ S_{\tilde{Y}^{(r+1)}}(\phi_r -, \phi_r -),$$

where $S_{\tilde{Y}^{(r+1)}}$ is the standard pairing on $\omega_{\tilde{Y}^{(r+1)}}$. Therefore, the k-th hypercohomology of the de Rham complex $\mathrm{DR}_X \mathcal{P}_r$ is isomorphic to $H^{n-r+k}(\tilde{Y}^{(r+1)}, \mathbb{C})(-r)$ as polarized Hodge structures of weight $n+r+k$. Summing all the hypercohomologies of $\mathrm{DR}_X\mathcal{P}_r$, we get a polarized Hodge-Lefschetz structure of central weight $n+r$ with $\mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$. For the case when Y is non-reduced, we will identify the primitive part $\mathcal{P}_{\alpha,r}$ with certain filtered holonomic $\mathscr{D}$-modules coming from the cyclic coverings (Theorem 7.13), and the identification also respects the sesquilinear pairing (Theorem 8.10). As a direct consequence, we obtain

Theorem E. *Let $V^{\alpha}_{\ell,k} = H^{\ell}(X, \mathrm{gr}^W_k \mathrm{DR}_X\mathcal{M}_\alpha)$ be the relabelling of the first page of the weight spectral sequence. Then $V^\alpha = \bigoplus_{k,\ell\in\mathbb{Z}} V^\alpha_{\ell,k}$ is a polarized bigraded Hodge-Lefschetz structure of central weight n with the polarization induced by S_α and $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$ and R_α. Moreover, the differential d_1 of the first page of weight spectral is a differential of polarized bigraded Hodge-Lefschetz structure.*

By a formal argument of Guillén and Navarro Aznar [GNA90], which follows some ideas of Deligne and Saito, we have

Corollary F. *We have the following statements:*

(1) the Hodge spectral sequence degenerates at ${}^F E_1$;
(2) the weight spectral sequence degenerates at ${}^W E_2$;
(3) the α-generalized eigenspace of the bigraded vector space ${}^W E_2 = \bigoplus_{\ell,k\in\mathbb{Z}} \mathrm{gr}^W_\ell H^k(Y, \Omega^\bullet_{X/\Delta}(\log Y)|_Y)$ with respect to R is a polarized bigradged Hodge-Lefschetz structure of central weight n with polarization induced by S_α and $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$ and R_α.

Note that the third statement in the above Corollary is equivalent to the Theorem A; therefore, we finish the proof of Theorem A. See Theorem 6.6 and Corollary 6.7, when Y is reduced. See Theorem 8.11 and Corollary 8.12, when Y is allowed to be non-reduced,.

1.4. **Outline.** We first review basic notions on holonomic filtered $\mathscr{D}$-modules, integrable logarithmic connections and polarized bigraded Hodge-Lefschetz structures in §2. Then we set up the relative log de Rham complex and construct a log connection on its higher direct images in §3. We transfer all of the data on the relative log de Rham complex into a filtered holonomic $\mathscr{D}$-module in §4. To avoid the messy calculations, we first prove everything in the reduced case in §5 and §6. The idea for the non-reduced case is almost the same but requires some Hodge theory of cyclic coverings. We construct some $\mathscr{D}$-modules in §7.4 as the summand of the primitive part and prove their hypercohomogies underlies canonical polarized Hodge structures in §8.1. Lastly, we prove the local invariant cycle theorem in §9.

1.5. **Acknowledgement.** The author thanks his advisor Christian Schnell for introducing this topic to the author, and also for sharing ideas and discussing details during our weekly meetings. Many ideas of this paper should be attributed to him. The author also would like to thank Guodu Chen and Nathan Chen for reading a draft of this paper and useful discussions and Yilong Zhang for pointing out some typos in the earlier versions of the paper.

2. Preliminaries

2.1. **Filtered $\mathscr{D}$-modules with sesquilinear pairings.** We will work with right $\mathscr{D}$-modules unless further specified. Let Z be a complex manifold of dimension n and denote by Ω^p_Z the sheaf of holomorphic p-forms and $\mathscr{T}_Z$ the sheaf of holomorphic tangent vectors fields. For a filtered $\mathscr{D}_Z$-module we mean a pair $(\mathcal{N}, F_\bullet\mathcal{N})$ where $\mathcal{N}$ is a coherent $\mathscr{D}_Z$-module and $F_\bullet\mathcal{N}$ is a good filtration. Occasionally we will abuse notations and say $\mathcal{N}$ also denotes the filtered $\mathscr{D}_Z$-module if the filtration is clear. Denote by $\mathrm{gr}^F\mathscr{D}_Z = \bigoplus_{\ell\in Z}\mathrm{gr}^F_\ell\mathscr{D}_X$ the associated graded algebra and $\mathrm{gr}^F\mathcal{N} = \bigoplus_{\ell\in\mathbb{Z}}\mathrm{gr}^F_\ell\mathcal{N}$ the associated graded module. Note that $\mathrm{gr}^F\mathcal{N}$ is a coherent $\mathrm{gr}^F\mathscr{D}_Z$-module. Let $T^*Z = \mathrm{Spec}_Z\mathrm{gr}^F\mathscr{D}_X$ be the algebraic cotangent bundle and $T^*_V Z$ the geometric conormal bundle of a subvariety V

in Z. The *characteristic variety* of $\mathcal{N}$ is the support of $\mathrm{gr}^F\mathcal{N}$ on T^*Z and is denoted by $char(\mathcal{N})$. The *characteristic cycle* of $\mathcal{N}$ is the cycle associated to the coherent sheaf $\mathrm{gr}^F\mathcal{N}$ on T^*Z and is denoted by $cc(\mathcal{N})$. Neither the characteristic variety nor the characteristic cycle depend on the choice of the filtration [HTT08]. For example, the canonical bundle ω_Z is naturally a holonomic $\mathscr{D}_Z$-module with action

$$\alpha.\xi = -d(\xi \lrcorner \alpha)$$

for local sections $\xi \in \mathscr{T}_Z$ and $\alpha \in \omega_Z$. It also naturally has a good filtration

$$F_\ell\omega_Z = \begin{cases} \omega_Z, & \ell \geq -n; \\ 0, & \ell < -n. \end{cases} \tag{2.1}$$

Then one can compute $cc(\omega_Z) = [T^*_Z Z]$ which is the cycle of the zero section of the cotangent bundle. We call $\mathcal{N}$ a *holonomic* $\mathscr{D}_Z$-module if $\dim char(\mathcal{N}) = n$. See more details in [HTT08]. A *Tate twist* of filtered $\mathscr{D}_Z$-module is defined to be $\mathcal{N}(-r) = (\mathcal{N}, F_{\bullet+r}\mathcal{N})$ for any $r \in \mathbb{Z}$.

Denote by $\mathbf{D}^b(Z,\mathbb{C})$ the bounded derived category of complexes with values in finite dimensional $\mathbb{C}$-vector spaces and $\mathbf{D}^b(Z,\mathscr{D})$ the bounded derived category of $\mathscr{D}_Z$-modules. Denote by $\mathbf{D}^b_h(Z,\mathscr{D})$ the full subcategory of $\mathbf{D}^b(Z,\mathscr{D})$ whose objects are complexes with holonomic cohomologies. For a morphism $f: Z \to W$ between complex manifolds, denote by $Rf_*, Rf_! : \mathbf{D}^b(Z,\mathbb{C}) \to \mathbf{D}^b(W,\mathbb{C})$ the derived pushforward and proper pushforward functors respectively and $R^kf_*, R^kf_!$ the k-th cohomology functors respectively. For any $\mathcal{N}^\bullet \in \mathbf{D}^b(Z,\mathscr{D})$, the pushforward functor and the proper pushfoward functor $f_+, f_\dagger : \mathbf{D}^b(Z,\mathscr{D}) \to \mathbf{D}^b(W,\mathscr{D})$ are by definition, respectively

$$f_+\mathcal{N}^\bullet = Rf_*(\mathcal{N}^\bullet \overset{L}{\underset{\mathscr{D}_Z}{\otimes}} \mathscr{D}_{Z\to W}) \text{ and } f_\dagger\mathcal{N}^\bullet = Rf_!(\mathcal{N}^\bullet \overset{L}{\underset{\mathscr{D}_Z}{\otimes}} \mathscr{D}_{Z\to W}),$$

where $\mathscr{D}_{Z\to W} = f^*\mathscr{D}_W$ is the transfer module. In fact, the functor $f_\dagger$ preserves the holonomicty, i.e., $f_\dagger : \mathbf{D}^b_h(Z,\mathscr{D}) \to \mathbf{D}^b_h(W,\mathscr{D})$ (see [HTT08]). Of course if f is proper or proper on the support of $\mathcal{N}$ then $f_+ = f_\dagger$. The *de Rham complex* of $\mathcal{N}$ is

$$\mathrm{DR}_Z\mathcal{N} =_{\mathrm{def}} \mathcal{N} \otimes \bigwedge^{-\bullet}\mathscr{T}_Z = \{\mathcal{N} \otimes \bigwedge^{n}\mathscr{T}_Z\mathcal{N} \to \mathcal{N} \otimes \bigwedge^{n-1}\mathscr{T}_Z \to \cdots \to \mathcal{N}\}$$

with $\mathcal{N}$ is in degree 0. If without further indication, tensor products are always taken over $\mathscr{O}$-modules. Some authors also call it Spencer complex. The de Rham complex of ω_Z

$$\omega_Z \otimes \bigwedge^{-\bullet}\mathscr{T}_Z = \{\omega_Z \otimes \bigwedge^{n}\mathscr{T}_Z\omega_Z \to \omega_Z \otimes \bigwedge^{n-1}\mathscr{T}_Z \to \cdots \to \omega_Z\}$$

is isomorphic to the usual de Rham complex $\mathrm{DR}_Z\mathscr{O}_Z = \Omega_Z^{n+\bullet}$ of Z under the isomorphisms

$$\omega_Z \otimes \bigwedge^{p}\mathscr{T}_Z \to \Omega_Z^{n-p},\ \omega \otimes \partial_J \mapsto (-1)^{n-j_1+\cdots+n-j_p}dz_{\bar{J}}, \tag{2.2}$$

where ∂_J is a local section of $\bigwedge^p\mathscr{T}_Z$, J is ordered index set and $\bar{J}$ is the complement with the natural ordering, and $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$. If $F_\bullet\mathcal{N}$ is a good filtration, the de Rham complex is also filtered:

$$F_\ell\mathrm{DR}_Z\mathcal{N} = F_{\ell+\bullet}\mathcal{N} \otimes \bigwedge^{-\bullet}\mathscr{T}_Z = \{F_\ell\mathcal{N} \otimes \bigwedge^{n}\mathscr{T}_Z\mathcal{N} \to F_{\ell+1}\mathcal{N} \otimes \bigwedge^{n-1}\mathscr{T}_Z \to \cdots \to F_{\ell+n}\mathcal{N}\}.$$

The direct image functor and the de Rham functor are commute : $Rf_! \circ \mathrm{DR}_Z = \mathrm{DR}_W \circ f_\dagger$ [MS, Corollary 4.4.4].

A *sesquilinear pairing* S on $\mathscr{D}_Z$-module $\mathcal{N}$ is a $\mathscr{D}_{Z,\overline{Z}}$-module morphism $S: \mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_Z$. Here, $\mathscr{D}_{Z,\overline{Z}} = \mathscr{D}_Z \otimes_{\mathbb{C}} \mathscr{D}_{\overline{Z}}$ for $\overline{\mathscr{D}_Z}$ is the sheaf antiholomorphic differential operators, $\overline{\mathcal{N}}$ is the stupid conjugate of $\mathcal{N}$ as a $\overline{\mathscr{D}}_Z$-module and $\mathfrak{C}_Z$ is the sheaf of currents on Z with natural $\mathscr{D}_{Z,\overline{Z}}$-module structure. We have the proper pushforward functor similarly as above on $\mathscr{D}_{Z,\overline{Z}}$-modules and also call it $f_\dagger$:

$$f_\dagger(-) =_{\mathrm{def}} Rf_!(- \overset{L}{\underset{\mathscr{D}_{Z,\overline{Z}}}{\otimes}} \mathscr{D}_{Z,\overline{Z}\to W,\overline{W}}),$$

where the transfer module $\mathscr{D}_{Z,\overline{Z}\to W,\overline{W}} =_{\mathrm{def}} f^*\mathscr{D}_{W,\overline{W}}$. Because of the natural morphism $f_\dagger\mathfrak{C}_Z \to \mathfrak{C}_W$, we can pushforward the sesquilinear pairing to get

$$\mathscr{H}^0 f_\dagger S_k : \mathscr{H}^k f_\dagger\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathscr{H}^{-k} f_\dagger\mathcal{N}} \to \mathscr{H}^0 f_\dagger\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_W.$$

If f is a closed embedding then $f_+S : f_+\mathcal{N} \otimes_{\mathbb{C}} f_+\overline{\mathcal{N}} \to \mathfrak{C}_W$. If W is a point, then we have an induced pairing on the complex

$$f_\dagger S : \mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z \simeq \mathbb{C}[2n],$$

where $\mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \simeq \mathrm{DR}_Z\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathrm{DR}_Z\mathcal{N}}$. Taking cohomology at 0-th degree yields, for each $k \in \mathbb{Z}$,

$$H^k_c(Z, \mathrm{DR}_Z\mathcal{N}) \otimes H^{-k}_c(Z, \overline{\mathrm{DR}_Z\mathcal{N}}) \to H^0_c(Z, \mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}}) \to H^{2n}_c(Z, \mathbb{C}) \simeq \mathbb{C}. \tag{2.3}$$

Example 2.1. The $\mathscr{D}_Z$-module ω_Z carries a natural pairing $S_Z : \omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z} \to \mathfrak{C}_Z$,

$$\langle S_Z(m', m''), \eta\rangle = \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_Z \eta m' \wedge \overline{m''}, \tag{2.4}$$

for m', m'' local sections of ω_Z, η a test function on Z and $\varepsilon(k) = (-1)^{\frac{k(k-1)}{2}}$. The coefficient $\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n}$ in the definition is chosen so that $\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} m \wedge \overline{m} = |m|^2$ is a positive current for any local section m of ω_Z and elimination the choice of orenation (see more details in §2.3). The pairing $S_Z : \omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z} \to \mathfrak{C}_Z$ yields a collection of pairings

$$H^k_c(Z, \mathrm{DR}_Z\omega_Z) \otimes_{\mathbb{C}} \overline{H^{-k}_c(Z, \mathrm{DR}_Z\omega_Z)} \to \mathbb{C}.$$

2.2. **Logarithmic connections.** If $D = \sum a_i D_i$ is a simple normal crossing divisor on Z for $a_i \geq 0$, denote by $\Omega_Z(\log D)$ the sheaf of meromorphic differential 1-forms with logarithmic poles along $D_{\mathrm{red}} = \sum D_i$ and denote by $\Omega^p_Z(\log D) = \bigwedge^p \Omega_Z(\log D)$ the meromophic p-forms with logarithmic pole along D. Each $\Omega^p_Z(\log D)$ is a locally free $\mathscr{O}_Z$-module.

In our convention, the *de Rham complex* of Z is $\mathrm{DR}_Z\mathscr{O}_Z$

$$\Omega^{\bullet+n}_Z = \{\mathscr{O}_Z \to \Omega_Z \to \Omega^2_Z \to \cdots \to \Omega^n_Z\}[n].$$

The *log de Rham complex* is

$$\Omega^{\bullet+n}_Z(\log D) = \{\mathscr{O}_Z \to \Omega_Z(\log D) \to \Omega^2_Z(\log D) \to \cdots \to \Omega^n_Z(\log D)\}[n].$$

We will follow the Koszul sign rule: for a chain complex $C^\bullet$ with differential d, the shifted complex $C^{\bullet+n} = C^\bullet[n]$ equipped with differential $(-1)^n d$. We define residue along D_i by (see [EV92, 2.5])

$$\mathrm{Res}_{D_i} : \Omega^{\bullet+n}_Z(\log D) \to \Omega^{\bullet+\dim D_i}_{D_i}(\log(D - D_i)|_{D_i}), \ \frac{dz_i}{z_i} \wedge \alpha \mapsto \alpha|_{D_i},$$

where z_i is the local defining equation of D_i and $\frac{dz_i}{z_i} \wedge \alpha$ is a local section of $\Omega^{\bullet+n}_Z(\log D)$. It factors through

$$\Omega^{\bullet+n}_Z(\log D)|_{D_i} \to \Omega^{\bullet+\dim D_i}_{D_i}(\log(D - D_i)|_{D_i}).$$

By abuse of notations, we still call the above morphism Res_{D_i}. Let $D^J = \cap_{j\in J} D^J$ and $D_J = \sum_{j\in J} D_j$. Then we have a collection of residue maps, by choosing an order on the indices and successively applying Res_{D_j} for $j \in J$,

$$\mathrm{Res}_{D^J} : \Omega^{\bullet+n}_Z(\log D) \to \Omega^{\bullet+\dim D^J}_{D^J}(\log(D - D_J)|_{D^J}).$$

A *log connection* ∇ with poles along D on a coherent $\mathscr{O}_Z$-module $\mathcal{F}$ is a $\mathbb{C}$-linear morphism $\nabla : \mathcal{F} \to \Omega_Z(\log D) \otimes \mathcal{F}$ satisfying the Leibniz rule $\nabla fs = df \otimes s + f\nabla s$ for f local section of $\mathscr{O}_Z$ and s local section of $\mathcal{F}$. One can extend standardly ∇ to a complex

$$\mathcal{F} \xrightarrow{\nabla} \Omega_Z(\log D) \otimes \mathcal{F} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_Z(\log D) \otimes \mathcal{F}.$$

If the above is a chain complex, i.e., $\nabla^2 = 0$ we say $(\mathcal{F}, \nabla)$ is an *integrable* log connection. For any integrable log connection $\nabla : \mathcal{F} \to \Omega_Z(\log D) \otimes \mathcal{F}$, we call the morphism $\mathrm{Res}_{D_i}\nabla : \mathcal{F} \to \mathcal{F}|_{D_i}$ induced by $\mathrm{Res}_{D_i} : \Omega_Z(\log D) \to \mathscr{O}_{D_i}$ its *residue* along D_i. Note that Res_{D_i} is $\mathscr{O}_Z$-linear and factors through again $\mathcal{F}|_{D_i} \to \mathcal{F}|_{D_i}$.

An integrable log connection is same as a left $\mathscr{D}_Z(\log D)$-module, where $\mathscr{D}_Z(\log D)$ is the sub-algebra of $\mathscr{D}_Z$ generated locally by the differential operators P such that $P \cdot \mathscr{I}_D \subset \mathscr{I}_D$. Here, we denote by $\mathscr{I}_D$ the ideal sheaf of the normal crossing divisor D. Then we can extend the definition of residues of a log connection as follows. The sheaf $\mathscr{O}_{D_i} = \mathscr{O}_Z/\mathscr{I}_{D_i}$ naturally has a left $\mathscr{D}_Z(\log D)$-module structure because $\mathscr{I}_{D_i}$ is also stable under by the $\mathscr{D}_Z(\log D)$-action by the naive reason. Let $\mathcal{F}^\bullet$ be a complex of integrable log connections. Then the complex

$$\mathcal{F}^\bullet \overset{L}{\underset{\mathscr{O}_Z}{\otimes}} \mathscr{O}_{D_i}$$

is a complex of $\mathscr{D}_Z(\log D)$-modules because taking tensor products over $\mathscr{O}_Z$ is closed in the category of $\mathscr{D}_Z(\log D)$-modules and one can resolve either $\mathcal{F}^\bullet$ or $\mathscr{O}_{D_i}$ using locally $\mathscr{D}_Z(\log D)$-free resolutions. The ℓ-th cohomology $\mathscr{H}^\ell(\mathcal{F}^\bullet \otimes^L \mathscr{O}_{D_i})$ is indeed $\mathscr{O}_{D_i}$-module equipped with an integrable log connection. The residue of of this log connection is $\mathscr{O}_{D_i}$-linear and is called the the ℓ-th *residue* of the complex $\mathcal{F}^\bullet$.

As in the case of $\mathscr{D}$-module, the sheaf $\omega_Z(\log D) = \Omega_Z^n(\log D)$ carries a canonical right $\mathscr{D}_Z(\log D)$-module structure and we have the left to right transformation $\mathcal{F} \mapsto \omega_Z(\log D) \otimes \mathcal{F}$ for any left $\mathscr{D}_Z(\log D)$-module $\mathcal{F}$. Moreover, we have the following analog

Theorem 2.2. *The log de Rham complex of $\mathscr{D}_Z(\log D)$*

$$\{\mathscr{D}_Z(\log D) \to \Omega_Z(\log D) \otimes \mathscr{D}_Z(\log D) \to \cdots \to \Omega_Z^n(\log D) \otimes \mathscr{D}_Z(\log D)\}\,[n]$$

is a resolution of $\omega_Z(\log D)$ as right $\mathscr{D}_Z(\log D)$-modules. The Spencer complex of $\mathscr{D}_Z(\log D)$

$$\mathscr{D}_Z(\log D) \otimes \bigwedge^n \mathscr{T}_Z(\log D) \to \mathscr{D}_Z(\log D) \otimes \bigwedge^{n-1} \mathscr{T}_Z(\log D) \to \cdots \to \mathscr{D}_Z(\log D)$$

is a resolution of $\mathscr{O}_Z$ as left $\mathscr{D}_Z(\log D)$-modules.

For any integrable log connection $\mathcal{F}$, it induces a complex of right $\mathscr{D}_Z$-modules,

$$\{\mathcal{F} \otimes \mathscr{D}_Z \to \Omega_Z(\log D) \otimes \mathcal{F} \otimes \mathscr{D}_Z \to \cdots \to \Omega_Z^n(\log D) \otimes \mathcal{F} \otimes \mathscr{D}_Z\}[n]. \tag{2.5}$$

In fact, it is nothing but the log de Rham complex of $\mathcal{F} \otimes \mathscr{D}_Z$ as a left $\mathscr{D}_Z(\log D)$-module.

Lemma 2.3. *The log de Rham complex of $\mathcal{F} \otimes \mathscr{D}_Z$ is a $\mathscr{D}_Z$-module resolution of*

$$\omega_Z(\log D) \otimes \mathcal{F} \underset{\mathscr{D}_Z(\log D)}{\otimes} \mathscr{D}_Z.$$

Proof. By the above theorem, we have

$$\begin{aligned}
\omega_Z(\log D) \otimes \mathcal{F} \underset{\mathscr{D}_Z(\log D)}{\otimes} \mathscr{D}_Z &\simeq \omega_Z(\log D) \otimes \mathcal{F} \underset{\mathscr{D}_Z(\log D)}{\otimes} \left(\mathscr{D}_Z(\log D) \otimes \bigwedge^{-\bullet} \mathscr{T}_Z(\log D)\right) \otimes \mathscr{D}_Z \\
&= \omega_Z(\log D) \otimes \mathcal{F} \otimes \bigwedge^{-\bullet} \mathscr{T}_Z(\log D) \otimes \mathscr{D}_Z \\
&\simeq \Omega_Z^{\bullet+n}(\log D) \otimes \mathcal{F} \otimes \mathscr{D}_Z.
\end{aligned}$$

The last isomorphism follows from that the contraction $\omega_Z(\log D) \otimes \wedge^{-\bullet} \mathscr{T}_Z(\log D) \simeq \Omega_Z^{\bullet+n}(\log D)$. ☐

Example 2.4. We will use the following fact: the complex of right $\mathscr{D}_Z$-modules

$$\{\mathscr{D}_Z \to \Omega_Z(\log D) \otimes \mathscr{D}_Z \to \cdots \to \Omega_Z^n(\log D) \otimes \mathscr{D}_Z\}[n]$$

is a filtered resolution of $\omega_Z(*D) = \cup_{k\in\mathbb{Z}}\omega_Z(kD)$, equipped the induced filtration by $\Omega_Z^{n+\bullet}(\log D)\otimes F_{\ell+n+\bullet}\mathscr{D}_Z$. In fact, it is well-known that the inclusion $\Omega_Z^{n+\bullet}(\log D)\to\Omega_Z^{n+\bullet}(*D)$ is a filtered quasi-isomorphism [Del71]. The inclusion extends to a filtered quasi-isomorphism $\Omega_Z^{n+\bullet}(\log D)\otimes\mathscr{D}_Z\to\Omega_Z^{n+\bullet}(*D)\otimes\mathscr{D}_Z$. Since $\Omega_Z^{n+\bullet}(*D)\otimes\mathscr{D}_Z$ is a filtered resolution of $\omega_Z(*D)$, we conclude the proof. It follows that, for $f: Z\to W$,

$$f_\dagger\omega_Z(*D) = Rf_!(\omega_Z(*D)\otimes^L_{\mathscr{D}_Z}\mathscr{D}_{Z\to W}) = Rf_!\Omega_Z^{n+\bullet}(\log D)\otimes\mathscr{D}_W.$$

In particular, if f is a closed embedding then $f_! = f_+$ is right exact and $f_\dagger = \mathscr{H}^0 f_\dagger$, which means

$$\{\mathscr{D}_W\to f_+\Omega_Z(\log D)\otimes\mathscr{D}_W\to\cdots\to f_+\Omega_Z^n(\log D)\otimes\mathscr{D}_W\}[n]$$

is a resolution of $f_\dagger\omega_Z(*D)$. We put the induced filtration to make it a filtered resolution and denote by

$$f_\dagger(\omega_Z(*D), F_\bullet\omega_Z(*D)) = (f_\dagger\omega_Z(*D), F_\bullet f_\dagger\omega_Z(*D)),$$

or for simplicity just $f_\dagger\omega_Z(*D)$.

The $\mathscr{D}_Z$-module looks like $\mathcal{L}\otimes\mathscr{D}_Z$ for $\mathcal{L}$ is a $\mathscr{O}_Z$-module is called *induced* $\mathscr{D}_Z$-module. For example, we have seen $\Omega_Z^{\dim Z+\bullet}\otimes\mathscr{D}_Z$ and $\Omega(\log D)_Z^{\dim Z+\bullet}\otimes\mathscr{D}_Z$ are complexes of induced $\mathscr{D}_Z$-modules.

2.3. **Polarized Hodge-Lefschetz structures.** The goal of this subsection is to introduce polarized bigraded Hodge-Lefschetz structures. The prototype of polarized Hodge-Lefschetz structures one should keep in mind is the graded vector space consisting of cohomologies of a compact Kähler manifold. Polarized bigraded Hodge-Lefschetz structures are the degenerations of polarized Hodge-Lefschetz structures. We begin with the convention on Hodge structures and we only consider complex Hodge structures.

A *Hodge structure* of *weight* n is a finite dimensional vector space V with two decreasing filtrations $F^\bullet$ and $G^\bullet$ satisfying

$$V = F^p\oplus G^{n+1-p},$$

for each $p\in\mathbb{Z}$. Let $V^{p,q} = F^p\cap G^q$ for $p+q=n$. Then the above definition is equivalent to

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

A morphism of Hodge structures is just a morphism of vector spaces such that it preserves the two filtrations. A *polarization* on the Hodge structure $(V, F^\bullet, G^\bullet)$ is a non-degenerated hermitian pairing $S: V\otimes_{\mathbb{C}}\overline{V}\to\mathbb{C}$ such that

(1) F^p is orthogonal to G^{n+1-p} with respect to S for every $p\in\mathbb{Z}$;
(2) $(-1)^q S(-,-)$ is hermitian inner product on $V^{p,q}$.

Remark 2.5. A polarized Hodge structure of weight n is completely determined by the triple $(V, F_\bullet V, S)$ because

$$G^{n+1-p}V = \{a\in V : S(a,b) = 0 \text{ for all b in } F^pV\} = \overline{F^pV^{\perp_S}}.$$

We will also call the triple $(V, F_\bullet V, S)$ a polarized Hodge structure.

Remark 2.6. A Tate twist $(V, F^\bullet, S)(r)$ on a polarized Hodge structure $(V, F^\bullet, S)$ is the triple $(V, F^{\bullet+r}, (-1)^r S)$, for any integer r.

Now let us move on to the geometric case. It is well-known that the k-th cohomology group of a compact Kähler manifold Z has Hodge decomposition

$$H^k(Z,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Z)$$

and thus it is a Hodge structure of weight k. Fix a choice of $\sqrt{-1}$. Let Z be a compact Kähler manifold of dimension n, and let h be any Kähler metric on Z. We denote the Kähler form by $\omega = -\mathrm{Im}\, h\in A^2(Z,\mathbb{R})$ and denote its cohomology class by $[\omega] \in H^2(Z, \mathbb{R})$; note that this depends on the choice of $\sqrt{-1}$ through the function $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$. The choice of $\sqrt{-1}$ endows the two-dimensional real vector space $\mathbb{C}$ with an orientation on Z. The induced orientation on Z has the property that

$$\int_Z \frac{\omega^n}{n!} = \mathrm{vol}(Z) > 0.$$

The integral also depends on the orientation, hence on the choice of $\sqrt{-1}$. To remove the dependence, instead of the usual integral, we should use

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_Z : A^{2n}(Z, \mathbb{C}) \to \mathbb{C}.$$

Of course we still have

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_Z \frac{(2\pi\sqrt{-1}\omega)^n}{n!} = \mathrm{vol}(Z).$$

Let $L = [w]\wedge$ be the Lefschetz operator for a Kähler class $[w]$. Then for $k \le \dim Z$ the primitive part

$$P_L H^k(Z, \mathbb{C}) =_{\mathrm{def}} \ker L^{\dim Z - k} \cap H^k(X, \mathbb{C})$$

is a polarized Hodge structure of weight k with the polarization

$$S(a, b) = \frac{\varepsilon(n-k+1)}{(2\pi\sqrt{-1})^n} \int_Z (2\pi\sqrt{-1}L)^{n-k} a \wedge \bar{b},$$

for $a, b \in P_L H^k(Z, \mathbb{C})$ because of the Hodge-Riemman bilinear relation.

If we consider the cohomology groups all together, we will get the Hodge-Lefschetz strcuture of central weight n. Denote by $(\mathsf{X}, \mathsf{Y}, \mathsf{H})$ the $\mathfrak{sl}_2(\mathbb{C})$-triple, i.e.,

$$[\mathsf{X}, \mathsf{Y}] = \mathsf{H},\ [\mathsf{H}, \mathsf{X}] = 2\mathsf{X},\ [\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}.$$

In the Lie group $\mathrm{SL}_2(\mathbb{C})$, we have the Weil element $\mathsf{w} = e^{\mathsf{X}} e^{-\mathsf{Y}} e^{\mathsf{X}}$ with the property that $\mathsf{w}^{-1} = -\mathsf{w}$, and under the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on its Lie algebra, one has the identities

$$\mathsf{w}\mathsf{H}\mathsf{w}^{-1} = -\mathsf{H}, \quad \mathsf{w}\mathsf{X}\mathsf{w}^{-1} = -\mathsf{Y}, \quad \mathsf{w}\mathsf{Y}\mathsf{w}^{-1} = -\mathsf{X}$$

From this, one deduces that $e^{\mathsf{X}} = \mathsf{w} e^{-\mathsf{X}} e^{\mathsf{Y}} = e^{\mathsf{Y}} \mathsf{w} e^{\mathsf{Y}}$. Now $A^\bullet(Z)$ becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$ if we set

$$\mathsf{X} = 2\pi\sqrt{-1}L \quad \text{and} \quad \mathsf{Y} = (2\pi\sqrt{-1})^{-1}\Lambda$$

and let H act as multiplication by $k - n$ on the subspace $A^k(Z)$. The reason for this (non-standard) definition is that it makes the representation not depend on the choice of $\sqrt{-1}$. It is easy to see how w acts on primitive forms. Suppose that $\alpha \in A^{n-k}(Z)$ satisfies $\mathsf{Y}\alpha = 0$. Then $\mathsf{w}\alpha \in A^{n+k}(Z)$. If we now expand both sides of the identity

$$e^{\mathsf{X}}\alpha = e^{\mathsf{Y}} \mathsf{w} e^{\mathsf{Y}} \alpha = e^{\mathsf{Y}} \mathsf{w}\alpha$$

into power series, and then compare terms in degree $n + k$, we get

$$\mathsf{w}\alpha = \frac{\mathsf{X}^k}{k!}\alpha.$$

This formula is the reason for using w (instead of the otherwise w^{-1}): there is no sign on the right-hand side.

A Hodge-Lefschetz structure is linear algebra data encoding both representation theoretic and Hodge theoretic information. Recall that a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation is a graded vector space $V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$ satisfying the following three equivalent conditions.

(1) each graded piece V_ℓ is the ℓ-eigenspace of H;
(2) the morphism $\mathsf{X}^\ell : V_{-\ell} \to V_\ell$ is an isomorphism for each $\ell \ge 0$;

(3) the morphism $\mathsf{Y}^\ell : V_\ell \to V_{-\ell}$ is an isomorphism for each $\ell \geq 0$.

Example 2.7. For any finite dimensional vector space V together with a nilpotent operator N, there exists a so-called monodromy filtration $W_\bullet$ uniquely determined by the following two conditions

- for each $\ell \in \mathbb{Z}$, $N : W_\ell \to W_{\ell-2}$;
- the induced operator $N^\ell : \mathrm{gr}^W_\ell \to \mathrm{gr}^W_{-\ell}$ is an isomorphism for each $\ell \geq 0$.

Let $\mathrm{gr}^W = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}^W_\ell$. The ℓ-th primitive part $P_N \mathrm{gr}^W_\ell = \ker N^{\ell+1} \cap \mathrm{gr}^W_\ell$ consists of the classes of generators of cyclic subspaces of V of dimension ℓ as $\mathbb{C}[N]$-modules for $\ell \geq 0$. For each generator v, we have $N^{\ell+1}v = 0$ but $N^\ell v \neq 0$ and also v is not a image of N. Therefore, we have the identification

$$P_N \mathrm{gr}^W_\ell = \frac{\ker N^{\ell+1}}{\ker N^\ell + \mathrm{im}\, N \cap \ker N^{\ell+1}}.$$

Furthermore, we have the Lefschetz decomposition $\mathrm{gr}^W_\ell = \bigoplus_{k\geq 0} N^k P_N V_{\ell+2k}$. Taking $N = \mathsf{Y}$, the Lefschetz structure and the grading uniquely determines the operator X such that $(\mathsf{X},\mathsf{Y},\mathsf{H})$ is a $\mathfrak{sl}_2(\mathbb{C})$-triple by the relation $\mathsf{X}\mathsf{Y}^k = k(\ell - k + 1)\mathsf{Y}^{k-1}$ on $P_N \mathrm{gr}^W_\ell$. Thus gr^W naturally is a representation of $\mathfrak{sl}_2(\mathbb{C})$.

By Hard Lefschetz theorem, for any compact Kähler manifold the vector space $\bigoplus_{\ell\in\mathbb{Z}} H^{\dim Z+\ell}(Z,\mathbb{C})$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$ by setting $\mathsf{X} = 2\pi\sqrt{-1}L$ the Lefschetz operator, $\mathsf{Y} = (2\pi\sqrt{-1})^{-1}\Lambda$ the adjoint operator. But because of the Lefschetz operator of is of type $(1,1)$, we actually have $\mathsf{X} : H^k(Z,\mathbb{C}) \to H^{k+1}(Z,\mathbb{C})(1)$ is a morphism of Hodge structures and $\mathsf{X}^\ell : H^{\dim Z-\ell}(Z,\mathbb{C}) \to H^{\dim Z+\ell}(Z,\mathbb{C})(\ell)$ is an isomorphism of Hodge structures. This leads to the following definition: a *Hodge-Lefschetz* structure of *central weight* n is a $\mathfrak{sl}_2(\mathbb{C})$-representation $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$ with two filtrations $F^\bullet V$ and $G^\bullet V$ such that

(1) each graded piece $(V_\ell, F^\bullet V_\ell, G^\bullet V_\ell)$ is a Hodge structure of weight $n+\ell$;
(2) the operator $\mathsf{X} : (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell) \to (V_{\ell+2}, F^{\bullet+1}V_{\ell+2},, G^{\bullet+1}V_{\ell+2})$ is a morphism of Hodge structures such that

$$\mathsf{X}^\ell : (V_{-\ell}, F^\bullet V_{-\ell}, G^\bullet V_{-\ell}) \to (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell)(\ell)$$

is an isomorphism of Hodge structures;
(3) the operator $\mathsf{Y} : (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell) \to (V_{\ell-2}, F^{\bullet-1}V_{\ell-2}, G^{\bullet-1}V_{\ell-2})$ is a morphism of Hodge structures such that

$$\mathsf{Y}^\ell : (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell) \to (V_{-\ell}, F^\bullet V_{-\ell}, G^\bullet V_{-\ell})(-\ell)$$

is an isomorphism of Hodge structures.

It follows from the definition the primitive part $P_X V_\ell$ is a sub-Hodge structure for each $\ell < 0$. Let $V_\ell = H^{\dim Z+\ell}(Z,\mathbb{C})$ and $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$. It follows that V is a Hodge-Lefschetz structure of central weight $\dim Z$. Hodge-Lefschetz structure interplays well with the Hodge-Riemann bilinear relation. A *polarization* on a Hodge-Lefschetz structure V of central weight n is a hermitian symmetric paring $S : V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that

(1) the restriction $S|_{V_\ell \otimes_{\mathbb{C}} \overline{V_{-k}}}$ is zero for $\ell + k \neq 0$;
(2) $S(\mathsf{X}-,-) = S(-,\mathsf{X}-)$ and $S(-,\mathsf{Y}-) = S(\mathsf{Y}-,-)$;
(3) $S_{-\ell}(\mathsf{X}^\ell-,-)$ is a polarization on $P_X V_{-\ell}$, or equivalently, $S_\ell \circ (\mathrm{id} \otimes \mathsf{w})$ is a polarization on V_ℓ where $S_\ell : V_\ell \otimes \overline{V_{-\ell}} \to \mathbb{C}$ is the restriction of S.

Note that $\mathsf{w} : V_k \to V_{-k}(-k)$ is automatically an isomorphism of Hodge structures (of weight $n+k$). We first prove an auxiliary formula. Suppose that $a \in V_{-\ell}$ is primitive, in the sense that $\mathsf{X}^{\ell+1}a = 0($ and $\ell \geq 0)$. Then $\mathsf{Y}a = 0$, and from $\mathsf{w}e^{-\mathsf{X}} = e^{\mathsf{X}}e^{-\mathsf{Y}}$, we get $\mathsf{w}e^{-\mathsf{X}}a = e^{\mathsf{X}}a$, and after expanding and comparing terms in degree $\ell - 2j$, also

$$\mathsf{w}\frac{\mathsf{X}^j}{j!}a = (-1)^j \frac{\mathsf{X}^{\ell-j}}{(\ell-j)!}a \tag{2.6}$$

since w^2 acts on $V_{-\ell+2j}$ as $(-1)^{-\ell+2j} = (-1)^\ell$, this formula is actually symmetric in j and $\ell - j$,.

Lemma 2.8. *If V is a Hodge-Lefschetz structure, then $\mathsf{w} : V_k \to V_{-k}(-k)$ is an isomorphism of Hodge structures.*

Proof. Any $a \in V_k$ has a unique Lefschetz decomposition

$$a = \sum_{j \geq \max(k,0)} \frac{\mathsf{X}^j}{j!} a_j$$

where $a_j \in V_{k-2j}$ satisfies $Y a_j = 0$. (We only need to consider $j \geq k$ in the sum because $\mathsf{X}^{2j-k+1} a_j = 0$, which implies that $\mathsf{X}^j a_j = 0$ for $j < k$.) Suppose further that $a \in V_k^{p,q}$, where $p + q = n + k$. Then $\mathsf{X}^i a_j \in V_{k+2i}^{p+i,q+i}$, and by descending induction on $j \geq \max(k, 0)$, we deduce that $a_j \in V_{k-2j}^{p-j,q-j}$. In other words, the Lefschetz decomposition holds in the category of Hodge structures.

We can now check what happens when we apply w. Using (2.6), we find that

$$\mathsf{w}a = \sum_{j \geq \max(k,0)} \mathsf{w}\frac{\mathsf{X}^j}{j!} a_j = \sum_{j \geq \max(k,0)} (-1)^j \frac{\mathsf{X}^{j-k}}{(j-k)!} a_j \in V_{-k}^{p-k,q-k}$$

and so w is a morphism of Hodge structures. The same calculation shows that w^{-1} is also a morphism of Hodge structures. It follows that w is an isomorphism of Hodge structures. ☐

The definition of polarized Hodge-Lefschetz structure of central weight n is redundant. In fact the definition is equivalent to a tuple $(V, \mathsf{X}, F^\bullet, S)$ for $V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$, $F^\bullet$ is a decreasing filtration, $\mathsf{X} : (V_\ell, F^\bullet) \to (V_{\ell+2}, F^{\bullet+1})$, and S is a Hermitian pairing such that

(pHL1) for each $\ell \geq 0$, $\mathsf{X}^\ell : F^\bullet V_{-\ell} \to F^{\bullet+\ell} V_\ell$ is an isomorphism;
(pHL2) $S(\mathsf{X}-,-) = S(-,\mathsf{X}-)$ and $S|_{V_\ell \otimes_{\mathbb{C}} \overline{V_{-k}}}$ vanishes except for $k = -\ell$;
(pHL3) the triple $\left(P_{\mathsf{X}} V_j, F_\bullet, S \circ \left(\mathsf{X}^j \circ \mathrm{id}\right)\right)$ is a porlarized Hodge structure of weight $n - j$.

The condition (pHL1) in the above definition indicates the Lefschetz decomposition respects the filtration $F^\bullet$. Therefore Y is determined uniquely and also filtered. The second condition implies that $S(\mathsf{Y}-,-) = S(-,\mathsf{Y}-)$. The third condition says that $S \circ (\mathrm{id} \otimes \mathsf{w})$ is non-degenerate on $F^p V_\ell \otimes \overline{F^p V_{-\ell}}$. Therefore, we also get the following concrete description of the Hodge structure on V_ℓ: for $p + q = n + \ell$

$$V_\ell^{p,q} = \{a \in F^p V_\ell : S_\ell(a,b) = 0 \text{ for all } b \in F^{p-\ell+1} V_{-\ell}\},$$
$$G^q V_\ell = \{a \in V_\ell, S_\ell(a,b) = 0 \text{ for all } b \in F^{n-q+1} V_{-\ell}\}.$$

Example 2.9. For a compact Kähler manifold Z of dimension n, let $V_\ell = H^{n+\ell}(Z, \mathbb{C})$ and $V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$. Then V together with $\mathsf{X} = 2\pi\sqrt{-1}L$ and $\mathsf{Y} = (2\pi\sqrt{-1})^{-1}\Lambda$ and with the natural filtration is a Hodge-Lefschetz structure of central weight n. By Hodge-Riemann bilinear relation, taking

$$S_\ell(a,b) = \frac{\varepsilon(n+\ell+1)}{(2\pi\sqrt{-1})^n} \int_Z a \wedge \bar{b} = \varepsilon(\ell)(-1)^{\ell n} \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_Z a \wedge \bar{b} \tag{2.7}$$

for $a \in V_\ell$ and $b \in V_{-\ell}$ gives a polarization on V. The polarized Hodge-Lefschetz structure V is determined by the filtered $\mathscr{D}_Z$-module ω_Z together with the sesquilinear pairing S_Z. The graded piece V_ℓ is just ℓ-th hypercohomology of $\mathrm{DR}_Z \omega_Z$ with induced filtration $F^\bullet V_\ell$ given by the image of $H^\ell(Z, F_{-\bullet} \mathrm{DR}_Z \omega_Z)$. And the polarization S_k is given by $\varepsilon(k)$ times the pairing

$$H^k(Z, \mathrm{DR}_Z \omega_Z) \otimes \overline{H^{-k}(Z, \mathrm{DR}_Z \omega_Z)} \longrightarrow H^0(Z, \mathrm{DR}_{Z,\overline{Z}} \omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z}) \xrightarrow{S_Z} H^0(Z, \mathrm{DR}_{Z,\overline{Z}} \mathfrak{C}_Z) \simeq \mathbb{C}$$

We can work out the pairing explicitly. Note that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{DR}_{Z,\overline{Z}}\omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z} & \longrightarrow & \mathrm{DR}_{Z,\overline{Z}}\mathscr{O}_Z \otimes_{\mathbb{C}} \overline{\mathscr{O}_Z} \\ \downarrow S & & \downarrow D \\ \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z & \longrightarrow & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{Db}_Z \end{array}$$

where the upper horizontal arrow is the isomorphism induced by (2.2) and similarly the lower horizontal arrow is defined on the terms in degree $-k$,

$$\mathfrak{C}_Z \otimes_{\mathscr{O}_{Z,\bar{Z}}} \bigwedge^k \mathscr{T}_{Z,\bar{Z}} \to \Omega^{2n-k}_{Z,\bar{Z}} \otimes_{\mathscr{O}_{Z,\bar{Z}}} \mathfrak{Db}_Z$$

by the following rule: write a current locally as $D\omega \wedge \bar{\omega}$, with a distribution D and denote by $\partial_J = \bigwedge_J \partial_j$ and $dx_{\bar{J}} = \bigwedge_{i \notin J} dx_i$ for an ordered index subset J of I; then

$$(D\omega \wedge \bar{\omega}) \otimes \partial_J \wedge \bar{\partial}_K \mapsto (-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)}(-1)^{nq} dx_{\bar{J}} \wedge \overline{dx}_{\bar{K}} \otimes D \tag{2.8}$$

where $\#J = p$ and $\#K = q$, and $p + q = k$. The sign factor is explained by the number of swaps that are needed to move everything into the right place, which is $(2n - j_1) + \cdots + (2n - j_p) + (n - k_1) + \cdots + (n - k_q)$. We can now derive a formula for the induced pairing

$$\mathrm{DR}_Z \mathscr{O}_Z \otimes_{\mathbb{C}} \overline{\mathrm{DR}_Z \mathscr{O}_Z} \to \mathrm{DR}_{Z,\bar{Z}} \mathfrak{Db}_Z. \tag{2.9}$$

For the two local sections $\alpha = dx_{\bar{J}}$ and $\beta = dx_{\bar{K}}$, under the isomorphism $\mathrm{DR}_Z\mathscr{O}_Z \cong \mathrm{DR}_Z\omega_Z$ in (2.2), the $(n-p)$-form α goes to

$$(-1)^{np}(-1)^{j_1+\cdots+j_p} \cdot \omega \otimes \partial_J.$$

and the $(n-q)$-form β goes to

$$(-1)^{nq}(-1)^{k_1+\cdots+k_q} \cdot \omega \otimes \partial_K.$$

The pairing S_Z on $\mathrm{DR}_Z\omega_Z$ takes those two sections to

$$(-1)^{n(p+q)}(-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)} S(\omega, \omega) \otimes \partial_J \wedge \bar{\partial}_K \tag{2.10}$$

where S_Z is defined in (2.4). Now $S_Z(\omega, \omega) = D_Z\omega \wedge \bar{\omega}$, where D is the distribution

$$D_Z = \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_Z$$

Under the isomorphism in (2.8) the section (2.10) therefore goes to

$$(-1)^{np} dx_{\bar{J}} \wedge \overline{dx}_{\bar{K}} \otimes D_Z = (-1)^{n(\deg \alpha - n)} \alpha \wedge \bar{\beta} \otimes D_Z$$

The formula we have just derived also works for smooth forms, of course. In other words, the same formula can be used to extend (2.9) to a pairing on the de Rham complex of smooth forms. The resulting pairings on cohomology are, assuming Z is compact

$$H^{n+k}(Z, \mathbb{C}) \otimes \overline{H^{n-k}(Z, \mathbb{C})} \to \mathbb{C}, \quad (\alpha, \beta) \mapsto (-1)^{n(\deg \alpha - n)} \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_Z \alpha \wedge \bar{\beta}, \tag{2.11}$$

which coincides with the pairing (2.7) precisely.

2.4. **Polarized bigraded Hodge-Lefschetz structures.** In the paper, what we really consider is the degeneration of "variation of Hodge-Lefschetz structures" of a family of compact Kähler manifolds. As it turns out the limit of the degeneration is a bigraded Hodge-Lefschetz structure. We begin to define polarized bigraded Hodge-Lefschetz structures. Similarly to the case of $\mathfrak{sl}_2(\mathbb{C})$-representation, a $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-representation is a bigraded vector space $V=\bigoplus_{\ell,k\in\mathbb{Z}}V_{\ell,k}$ satisfying the following three equivalent conditions:

(1) each bigraded piece $V_{\ell,k}$ is the ℓ-th eigenspace of H_1 and k-th eigenspace of H_2;
(2) for each $\ell,k\in\mathbb{Z}$ we have $\mathsf{X}_1:V_{\ell,k}\to V_{\ell+2,k}$ and $\mathsf{X}_2:V_{\ell,k}\to V_{\ell,k+2}$ plus isomorphisms
$$\mathsf{X}_1^{\ell}:V_{-\ell,k}\to V_{\ell,k}\text{ and }\mathsf{X}_2^{k}:V_{\ell,-k}\to V_{\ell,k};$$
(3) for each $\ell,k\in\mathbb{Z}$ we have $\mathsf{Y}_1:V_{\ell,k}\to V_{\ell-2,k}$ and $\mathsf{Y}_2:V_{\ell,k}\to V_{\ell,k-2}$ plus the isomorphism
$$\mathsf{Y}_1^{\ell}:V_{\ell,k}\to V_{-\ell,k}\text{ and }\mathsf{Y}_2^{k}:V_{\ell,k}\to V_{\ell,-k}.$$

A *bigraded Hodge-Lefschetz structure* of *central weight* n is a $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-representation $V=\bigoplus_{\ell,k\in\mathbb{Z}}V_{\ell,k}$ with two filtrations $F^\bullet V$ and $G^\bullet V$ such that

(1) the bifiltered vector space $(V_{\ell,k},F^\bullet V_{\ell,k},G^\bullet V_{\ell,k})$ is a Hodge structure of weight $n+\ell+k$;
(2) the two operators $\mathsf{X}_1:(V_{\ell,k},F^\bullet,G^\bullet)\to(V_{\ell+2,k},F^{\bullet+1},G^{\bullet+1})$ and $\mathsf{X}_2:(V_{\ell,k},F^\bullet,G^\bullet)\to(V_{\ell,k+2},F^{\bullet+1},G^{\bullet+1})$ are morphisms of Hodge structures such that
$$\mathsf{X}_1^{\ell}:(V_{-\ell,k},F^\bullet,G^\bullet)\to(V_{\ell,k},F^\bullet,G^\bullet)(\ell)\quad\text{and}\quad\mathsf{X}_2^{k}:(V_{\ell,-k},F^\bullet,G^\bullet)\to(V_{\ell,k},F^\bullet,G^\bullet)(k)$$
are isomorphisms of Hodge structures.
(3) the two operators $\mathsf{Y}_1:(V_{\ell,k},F^\bullet,G^\bullet)\to(V_{\ell-2,k},F^{\bullet-1},G^{\bullet-1})$ and $\mathsf{Y}_2:(V_{\ell,k},F^\bullet,G^\bullet)\to(V_{\ell,k-2},F^{\bullet-1},G^{\bullet-1})$ are morphisms of Hodge structures such that
$$\mathsf{Y}_1^{\ell}:(V_{\ell,k},F^\bullet,G^\bullet)\to(V_{-\ell,k},F^\bullet,G^\bullet)(-\ell)\quad\text{and}\quad\mathsf{Y}_2^{k}:(V_{\ell,k},F^\bullet,G^\bullet)\to(V_{\ell,-k},F^\bullet,G^\bullet)(-k)$$
are isomorphisms of Hodge structures.

A *polarization* on a bigraded Hodge-Lefschetz structure $V=\bigoplus_{\ell,k\in\mathbb{Z}}V_{\ell,k}$ of central weight n is a hermitian symmetric pairing $S:V\otimes_{\mathbb{C}}\overline{V}\to\mathbb{C}$ such that

(1) the restriction $S|_{V_{\ell,k}\otimes_{\mathbb{C}}\overline{V_{i,j}}}:V_{\ell,k}\otimes_{\mathbb{C}}\overline{V_{i,j}}\to\mathbb{C}$ vanishies except for $\ell=-i$ and $k=-j$;
(2) $S(\mathsf{X}_1-,-)=S(-,\mathsf{X}_1-)$ and $S(-,\mathsf{Y}_2-)=S(\mathsf{Y}_2-,-)$;
(3) $S_{\ell,k}(\mathsf{X}_1^{\ell}-,(-\mathsf{Y}_2)^k-)$ is a polarization on the bi-primitive part $P_{-\ell,k}=\ker\mathsf{X}_1^{\ell+1}\cap\ker\mathsf{Y}_2^{k+1}\cap V_{-\ell,k}$, or equivalently, $S_{\ell,k}(-,\mathsf{w}_1\mathsf{w}_2-)$ is a polarization on $V_{\ell,k}$, where $S_{\ell,k}$ is the restriction of S on $V_{\ell,k}\otimes\overline{V_{-\ell,k}}$ and $\mathsf{w}_i=e^{\mathsf{X}_i}e^{-\mathsf{Y}_i}e^{\mathsf{X}_i}$ for $i=1,2$.

This is the practical definition because in the later application X_1 will be the $2\pi\sqrt{-1}L$ and Y_2 will be, up to a scalar, the logarithmic of the monodromy for the degeneration. Similiarly to the case of Hodge-Lefschetz structure, we have a simpler definition.

Theorem 2.10. *A polarized bigraded Hodge-Lefschetz structure of central weight n on a filtered bigraded vector space $(V=\bigoplus_{\ell,k}V_{\ell,k},F^\bullet V)$ is uniquely determined by the following:*

(pbHL1) *for every $\ell,k\in\mathbb{Z}$ we have two operators $\mathsf{X}_1:(V_{\ell,k},F^\bullet)\to(V_{\ell+2,k},F^{\bullet+1})$ and $\mathsf{Y}_2:(V_{\ell,k}F^\bullet)\to(V_{\ell,k-2},F^{\bullet-1})$ such that*
$$\mathsf{X}_1^{\ell}:F^\bullet V_{-\ell,k}\to F^{\bullet+\ell}V_{\ell,k}\quad\textit{and}\quad\mathsf{Y}_2^{k}:F^\bullet V_{\ell,k}\to F^{\bullet-k}V_{\ell,-k}\ \textit{are isomorphisms};$$
(pbHL2) *a collection of Hermitian pairings $S_{\ell,k}:V_{\ell,k}\otimes_{\mathbb{C}}\overline{V_{-\ell,-k}}\to\mathbb{C}$ such that*
$$S_{\ell,k}(\mathsf{X}_1-,-)=S_{\ell+2,k}(-,\mathsf{X}_1-)\quad\textit{and}\quad S_{\ell,k}(-,\mathsf{Y}_2-)=S_{\ell,k-2}(\mathsf{Y}_2-,-);$$

(pbHL3) *the triple* $\left(P_{-\ell,k}, F^\bullet P_{-\ell,k}, S\circ\left(\mathsf{X}_1^\ell\otimes(-\mathsf{Y}_2)^k\right)\right)$ *is a polarized Hodge structure of weight* $n-\ell+k$ *where* $F^\bullet P_{-\ell,k}=\ker \mathsf{X}_1^\ell\cap\ker\mathsf{Y}_2^k\cap F^\bullet V_{-\ell,k}$ *is the bi-primtive part.*

Then the Hodge structure on $V_{j,k}$ *can be described as: for* $p+q=n+j+k$

$$V_{j,k}^{p,q}=\{a\in F^pV_{j,k}:S_{j,k}(a,b)=0 \text{ for all } b\in F^{p-j-k+1}V_{-j-k}\},$$
$$G^qV_{j,k}=\{a\in V_{j,k}:S_{j,k}(a,b)=0 \text{ for all } b\in F^{n-q+1}V_{-j,-k}\}.$$

The proof is simple and is left to the reader. Later when we construct the limiting mixed Hodge structure, the polarized bigraded Hodge-Lefschetz structure naturally comes up from the first page of weight spectral sequence associated to a mixed Hodge complex. Modeled on the properties of the differential of spectral sequence we give the following definition:

A *differential* of a polarized bigraded Hodge Lefschetz structure $(V,F^\bullet,\mathsf{X}_1,\mathsf{Y}_2,S)$ is a linear map $d:V\to V$ such that

(1) $d:(V_{j,k},F^\bullet)\to(V_{j+1,k-1},F^\bullet)$ and $d^2=0$;
(2) d is skew-symmetrc with respect to S, i.e., $S(d-,-)+S(-,d-)=0$;
(3) $[\mathsf{X}_1,d]=0$ and $[\mathsf{Y}_2,d]=0$.

Remark 2.11. In fact, the above three conditions imply that d is a morphism of Hodge structures $d:V_{j,k}^{p,q}\to V_{j+1,k-1}^{p,q}$. A vector $a\in G^qV_{j,k}$ means that $S(a,b)=0$ for all $b\in F^{n-q+1}V_{-j,-k}$. Then $S(da,b)=S(a,db)=0$ for all $b\in F^{n-q+1}V_{-j-1,-k+1}$, indicating da belongs to $G^qV_{j+1,k-1}$.

The main result of this subsection is the following version of Deligne's lemma, showed by Guillén and Navarro Aznar.

Theorem 2.12 ([GNA90, (4.5)]). *The cohomology* $\ker d/\operatorname{im} d$ *of a polarized differential bigraded Hodge-Lefschetz struture is again a polarized bigraded Hodge-Lefschetz structure.*

Proof. Let $C:V\to V$ be the operator that acts as $(-1)^q$ on the subspace $V_{j,k}^{p,q}$ in the Hodge decomposition of each $V_{j,k}$. Since d is a morphism of Hodge structures, we have $[d,C]=0$. The fact that S is a polarization means that the Hermitian pairing

$$h^+:V\otimes_{\mathbb{C}}\overline{V}\to\mathbb{C},\quad h^+(a,b)=S\left(Ca,\mathsf{w}_1\mathsf{w}_2b\right)$$

is positive-definite on V. Let d^* be the adjoint of d with respect to h^+. Fix $a\in V_{j,k}$ and $b\in V_{j,k}$,:

$$\begin{aligned}h^+(da,b)&=S\left(Cda,\mathsf{w}_1\mathsf{w}_2b\right)=S\left(dCa,\mathsf{w}_1\mathsf{w}_2b\right)\\&=-S\left(Ca,d\mathsf{w}_1\mathsf{w}_2b\right)=-S\left(Ca,\mathsf{w}_1\mathsf{w}_2\cdot\mathsf{w}_2^{-1}\mathsf{w}_1^{-1}d\mathsf{w}_1\mathsf{w}_2\cdot b\right)=h^+\left(a,d^*b\right),\end{aligned}$$

i.e. the adjoint $d^*=-\mathsf{w}_2^{-1}\mathsf{w}_1^{-1}d\mathsf{w}_1\mathsf{w}_2$.

In addition to the two relations in the definition of differential

$$[\mathsf{X}_1,d]=0\quad\text{and}\quad[\mathsf{Y}_2,d]=0$$

we obtain from the grading another two relations

$$[\mathsf{H}_1,d]=d\quad\text{and}\quad[\mathsf{H}_2,d]=-d.$$

With respect to the $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-action on $\operatorname{End}_{\mathbb{C}}(V)$, the element d therefore has weight $(+1,-1)$, and is primitive with respect to the action by Y_1 and X_2. Define

$$d_1=[\mathsf{Y}_1,d]\quad\text{and}\quad d_2=-[\mathsf{X}_2,d].$$

The reason for the minus sign is that we have $[\mathsf{Y}_2, d] = 0$. Then d_1 has weight $(-1,-1)$, and is primitive with respect to the action by X_1 and X_2; this gives

$$\begin{aligned} &[\mathsf{H}_1, d_1] = -d_1, \quad [\mathsf{X}_1, d_1] = d, \qquad [\mathsf{Y}_1, d_1] = 0, \quad \mathsf{w}_1 d_1 \mathsf{w}_1^{-1} = d \\ &[\mathsf{H}_2, d_1] = -d_1, \qquad\qquad\qquad\quad [\mathsf{Y}_2, d_1] = 0. \end{aligned}$$

Similarly, d_2 has weight $(+1,+1)$, and therefore

$$\begin{aligned} &[\mathsf{H}_2, d_2] = d_2, \quad [\mathsf{X}_2, d_2] = 0, \quad [\mathsf{Y}_2, d_2] = -d, \quad \mathsf{w}_2 d_2 \mathsf{w}_2^{-1} = d \\ &[\mathsf{H}_1, d_2] = d_2, \quad [\mathsf{X}_1, d_2] = 0. \end{aligned}$$

Therefore, $d^* = -[\mathsf{Y}_1, d_2] = [\mathsf{X}_2, d_1] \in \operatorname{End}_{\mathbb{C}} V$. It has weight $(-1,+1)$, and is primitive with respect to X_1 and Y_2. From this, and the identities we already have, we deduce the following set of relations:

$$\begin{aligned} &[\mathsf{H}_1, d^*] = -d^*, \quad [\mathsf{X}_1, d^*] = d_2, \quad [\mathsf{Y}_1, d^*] = 0, \quad \mathsf{w}_1 d^* \mathsf{w}_1^{-1} = -d_2 \\ &[\mathsf{H}_2, d^*] = d^*, \quad [\mathsf{X}_2, d^*] = 0, \quad [\mathsf{Y}_2, d^*] = -d_1, \quad \mathsf{w}_2 d^* \mathsf{w}_2^{-1} = -d_1. \end{aligned}$$

We can check that the (formal) Laplace operator

$$\Delta = dd^* + d^* d \in \operatorname{End}_{\mathbb{C}}(V)$$

is invariant under the action of $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$. For example,

$$\begin{aligned} &[\mathsf{X}_1, dd^*] = \mathsf{X}_1 dd^* - dd^* \mathsf{X}_1 = d\mathsf{X}_1 d^* - d\,(\mathsf{X}_1 d^* + d_2) = -dd_2 \\ &[\mathsf{X}_1, d^* d] = \mathsf{X}_1 d^* d - d^* d\mathsf{X}_1 = (d^*\mathsf{X}_1 - d_2)\,d - d^*\mathsf{X}_1 d = -d_2 d \end{aligned}$$

from which we conclude, using $d^2 = 0$, that

$$[\mathsf{X}_1, \Delta] = -(dd_2 + d_2 d) = -(d\,(d\mathsf{X}_2 - \mathsf{X}_2 d) + (d\mathsf{X}_2 - \mathsf{X}_2 d)\,d) = 0$$

The other three commutators can be checked similarly. On the other hand, Δ is also a morphism of Hodge structures: the reason is that

$$d : V_{j,k} \to V_{j+1,k-1}, \quad \mathsf{Y}_1 : V_{j,k} \to V_{j-2,k}(-1), \quad \mathsf{X}_2 : V_{j,k} \to V_{j,k+2}(1)$$

are all morphisms of Hodge structures, and Δ is obtained by composing them in some order. It follows that $\ker \Delta \subseteq V$ is a bigraded Hodge-Lefschetz structure, polarized by the restriction of S. Because of the canonical isomorphism $\ker \Delta \simeq \ker d / \operatorname{im} d$ as bigraded Hodge-Lefschetz structures, the induced pairing by S on $\ker d/\operatorname{im} d$ is also a polarization. This concludes the proof. □

3. Log relative de Rham complex

Let $f : X \to \Delta$ be a proper holomorphic morphism smooth away from the origin whose central fiber Y is simple normal crossing but not necessarily reduced. Assume X is Kähler of dimension $n+1$ and $Y = \sum_{i \in I} e_i Y_i$ where Y_i's are smooth components and I a finite index set. Let t be a parameter on Δ and $z_0, z_1, z_2, \dots, z_n$ a local coordinate system on X such that $t = z_0^{e_0} z_1^{e_1} \cdots z_k^{e_k}$ such that $e_0, e_1, \dots, e_k \geq 1$. Then we have $\Omega_\Delta(\log 0) = \mathscr{O}_\Delta \cdot \frac{dt}{t}$ and $\Omega_X(\log Y)$ is locally generated by

$$e_0 \frac{dz_0}{z_0}, e_1 \frac{dz_1}{z_1}, \dots, e_k \frac{dz_k}{z_k}, dz_{k+1}, dz_{k+2}, \dots, dz_n$$

over $\mathscr{O}_X$. Denote by $\xi_0, \xi_1, \dots, \xi_n$ the image of the above generators in $\Omega_{X/\Delta}(\log Y)$, respectively. As a quotient of $\Omega_X(\log Y)$, the sheaf $\Omega_{X/\Delta}(\log Y)$ is generated by $\xi_0, \xi_1, \dots, \xi_n$, but under the relation

$$\xi_0 + \xi_1 + \cdots + \xi_n = 0 \quad \text{because} \quad f^* \frac{dt}{t} = e_0 \frac{dz_0}{z_0} + e_1 \frac{dz_1}{dz_1} + \cdots + e_k \frac{dz_k}{z_k}.$$

Let $\mathscr{T}_{X/\Delta}(\log Y)$ be the dual bundle of $\Omega_{X/\Delta}(\log Y)$. It is a subsheaf of $\mathscr{T}_X$, generated by

$$D_i = \begin{cases} \dfrac{1}{e_i} z_i \partial_i - \dfrac{1}{e_0} z_0 \partial_0, & 1 \le i \le k \\ \partial_i, & i > k, \end{cases} \tag{3.12}$$

where ∂_i is the local section of $\mathscr{T}_X$ dual to dz_i in Ω_X. It follows that $D_1, D_2, ..., D_n$ is the dual frame of $\xi_1, \xi_2, ..., \xi_n$.

3.1. **A "log connection".** We shall construct an operator in $\operatorname{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}\left(Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$ which should be regarded a "log connection". Note that we have the following short exact sequence of $\mathscr{O}_X$-modules

$$0 \to f^*\Omega_\Delta(\log 0) \otimes \Omega^{\bullet+n}_{X/\Delta}(\log Y) \to \Omega^{\bullet+n+1}_X(\log Y) \to \Omega^{\bullet+n+1}_{X/\Delta}(\log Y) \to 0.$$

Under the identification $\frac{dt}{t}\wedge : \mathscr{O}_X \to f^*\Omega_\Delta(\log 0)$, the above short exact sequence becomes

$$0 \longrightarrow \Omega^{\bullet+n}_{X/\Delta}(\log Y) \xrightarrow{\frac{dt}{t}\wedge} \Omega^{\bullet+n+1}_X(\log Y) \longrightarrow \Omega^{\bullet+n+1}_{X/\Delta}(\log Y) \longrightarrow 0.$$

Here, the morphism $\frac{dt}{t}\wedge : \Omega^k_{X/\Delta}(\log Y) \to \Omega^{k+1}_X(\log Y)$ works as $[\alpha] \mapsto \frac{dt}{t}\wedge\alpha$ which does not depend on the representative of $[\alpha]$. Let $\mathrm{Cone}^\bullet = \Omega^{\bullet+n}_X(\log Y) \oplus \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ be the mapping cone of $\frac{dt}{t}\wedge : \Omega^{\bullet+n-1}_{X/\Delta}(\log Y) \to \Omega^{\bullet+n}_X(\log Y)$. In our convention, the differential δ of the mapping cone works as $\delta(\alpha, [\beta]) = \left((-1)^n d\alpha + \frac{dt}{t}\wedge\beta, (-1)^n d[\beta]\right)$, where d is the usual exterior derivative on $\Omega^\bullet_X(\log Y)$ and by abuse of notation, also d denotes the induced differential on $\Omega^\bullet_{X/\Delta}(\log Y)$. Then we have the following diagram:

$$\begin{array}{ccc} \mathrm{Cone}^\bullet & \xrightarrow{\quad q \quad} & \Omega^{\bullet+n}_{X/\Delta}(\log Y) \\ \downarrow{\scriptstyle p} & \swarrow{\scriptstyle p\circ q^{-1}} & \\ \Omega^{\bullet+n}_{X/\Delta}(\log Y) & & \end{array} \tag{3.13}$$

where $q : \mathrm{Cone}^\bullet \to \Omega^{\bullet+n}_{X/\Delta}(\log Y), (\alpha, [\beta]) \mapsto [\alpha]$ is a quasi-isomorphism and p is the second projection. Therefore we have the morphism $p\circ q^{-1}$ in $\operatorname{End}_{\mathbf{D}^b(X,\mathbb{C})}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\right)$. For any local section $g \in \mathscr{O}_\Delta$, the multiplication by g is an endomorphism of $\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ because it is $f^{-1}\mathscr{O}_\Delta$-linear.

Lemma 3.1. *The operator $\nabla = (-1)^{n-1} p\circ q^{-1}$ satisfies $[\nabla, g] = tg'$ in $\operatorname{End}_{\mathbf{D}^b(X,\mathbb{C})}\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$, where g' denotes the derivative of $g \in \mathscr{O}_\Delta$.*

Proof. It is equivalent to show that $[p\circ q^{-1}, g] = (-1)^n tg'$. Define $g(\alpha, [\beta]) = (g\alpha, g[\beta] + (-1)^{n-1}tg'[\alpha])$ for any $(\alpha, [\beta]) \in \mathrm{Cone}^\bullet$ and $g \in f^{-1}\mathscr{O}_\Delta$. We shall show that g is an endomorphism of $\mathrm{Cone}^\bullet$, i.e., $g\delta(\alpha, [\beta]) = \delta g(\alpha, [\beta])$. This follows from that

$$\begin{aligned} g\delta(\alpha, [\beta]) &= g\left((-1)^n d\alpha + \frac{dt}{t}\wedge\beta, (-1)^n d[\beta]\right) \\ &= \left((-1)^n g d\alpha + g\frac{dt}{t}\wedge\beta, (-1)^n g d[\beta] - tg' d[\alpha]\right) \end{aligned}$$

and

$$\begin{aligned} \delta g(\alpha, [\beta]) &= \delta\left(g\alpha, g[\beta] + (-1)^{n-1}tg'[\alpha]\right) \\ &= \left((-1)^n dg\alpha + \frac{dt}{t}\wedge(g\beta + (-1)^{n-1}tg'\alpha), (-1)^n d(g[\beta] + (-1)^{n-1}tg'[\alpha])\right) \\ &= \left((-1)^n g d\alpha + g\frac{dt}{t}\wedge\beta, (-1)^n g d[\beta] - tg' d[\alpha]\right). \end{aligned}$$

It is easy to see that $g\circ q = q\circ g$ so that $q^{-1}\circ g = g\circ q^{-1}$. Therefore,

$$[p\circ q^{-1}, g] = p\circ q^{-1}\circ g - g\circ p\circ q^{-1} = [p,g]\circ q^{-1}$$

But $[p,g](\alpha,[\beta]) = p(g\alpha, g[\beta] + (-1)^{n-1}tg'[\alpha]) - g[\beta] = (-1)^{n-1}tg'[\alpha]$. It follows that

$$[p\circ q^{-1}, g]\circ q(\alpha,[\beta]) = [p,g](\alpha,[\beta]) = (-1)^{n-1}tg'\circ q(\alpha,[\beta]).$$

By inverse q we prove the statement. □

Because of the identification $\frac{dt}{t}\wedge : \mathscr{O}_\Delta \to \Omega_\Delta(\log 0)$, what we really get is a morphism in $\mathbf{D}^b(X,\mathbb{C})$

$$\nabla : \Omega^{\bullet+n}_{X/\Delta}(\log Y) \to f^*\Omega_\Delta(\log 0)\otimes \Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

such that $\nabla g = g\nabla + \frac{dt}{t}\otimes tg' \in \mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}(\Omega^{\bullet+n}_{X/\Delta}(\log Y))$ for any local section $g\in\mathscr{O}_\Delta$. Running the similar construction, we obtain an induced $\mathbb{C}$-linear (in fact $f^{-1}\mathscr{O}_\Delta$-linear) endomorphism $[\nabla]$ on $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ in $\mathbf{D}^b(X,\mathbb{C})$ satisfying the following diagram.

$$\begin{array}{ccccccc}
\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & \Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)[1] \\
\downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle [\nabla]} & & \downarrow{\scriptstyle (\nabla+1)[1]} \\
\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & \Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)[1]
\end{array}$$

Since $\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is $f^{-1}\mathscr{O}_\Delta$-linear, each cohomolgy $R^kf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is a coherent $\mathscr{O}_\Delta$-module. Taking direct image, we get $\mathbb{C}$-linear morphisms between distinguished triangles in $\mathbf{D}^b_{\mathrm{coh}}(\Delta,\mathscr{O}_\Delta)$:

$$\begin{array}{ccccccc}
Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)[1] \\
\downarrow{\scriptstyle Rf_*\nabla+1} & & \downarrow{\scriptstyle Rf_*\nabla} & & \downarrow{\scriptstyle Rf_*[\nabla]} & & \downarrow{\scriptstyle Rf_*(\nabla+1)[1]} \\
Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)[1]
\end{array} \tag{3.14}$$

where the morphism

$$Rf_*\nabla : Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \to Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

satisfies $[Rf_*\nabla, g] = tg' \in \mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}\left(Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$ for any local sections $g\in\mathscr{O}_\Delta$.

3.2. **Residue.** In the above situation, one should regard $Rf_*[\nabla]$ as the residue of $Rf_*\nabla$. More generally, let $\mathcal{F}^\bullet$ be a complex of $\mathscr{O}_\Delta$-modules with a morphism $\nabla\in\mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}(\mathcal{F}^\bullet)$ such that $[\nabla,g]=tg'$ for any $g\in\mathscr{O}_\Delta$. Let $\mathcal{G}^\bullet$ be the mapping cone of $t:\mathcal{F}^\bullet\to\mathcal{F}^\bullet$, which computes to $\mathcal{F}^\bullet\otimes^L\mathbb{C}(0)$. Then by the axioms of triangulated categories [HTT08], there exists an operator $R\in\mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}(\mathcal{G}^\bullet)$ making the following diagram commute in $\mathbf{D}^b(\Delta,\mathbb{C})$.

$$\begin{array}{ccccccc}
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{G}^\bullet = \mathcal{F}^\bullet\otimes^L\mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1] \\
\downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle (\nabla+1)[1]} \\
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{G}^\bullet = \mathcal{F}^\bullet\otimes^L\mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1]
\end{array}$$

We call the operator R a *residue* of ∇. Note that the axioms of triangulated categories cannot guarantee that the filling is unique. However, the eigenvalues of R_ℓ only depends on ∇, where R_ℓ denotes the induced operator on the cohomology $\mathscr{H}^\ell\left(\mathcal{F}^\bullet\otimes^L\mathbb{C}(0)\right)$. First, every object in $\mathbf{D}^b_{\mathrm{coh}}(\Delta,\mathscr{O})$ splits, meaning that $\mathcal{F}^\bullet\simeq\bigoplus_{\ell\in\mathbb{Z}}\mathscr{H}^\ell\mathcal{F}^\bullet[-\ell]$, since there are no Ext^i for $i\geq 2$ between two coherent sheaves over a curve. It follows that the morphism ∇ breaks up into sum of morphism consisting of diagonal morphism $\nabla_\ell:\mathscr{H}^\ell\mathcal{F}^\bullet[-\ell]\to\mathscr{H}^\ell\mathcal{F}^\bullet[-\ell]$ which is an actual log connection and off-diagonal morphism $\mathscr{H}^\ell \mathcal{F}^\bullet[-\ell] \to \mathscr{H}^m \mathcal{F}^\bullet[-m]$ but only for $\ell > m$. Thus the eigenvalues of R_ℓ are determined by ∇_ℓ and $\nabla_{\ell+1}$. When $\mathcal{F}^\bullet$ is a locally free sheaf centered at degree zero and ∇ is the usual log connection. Then above definition coincides with the usual definition of the residue of ∇.

Returning to our case, the natural choice of a residue of $Rf_*\nabla$ is $R = Rf_*[\nabla]$ because of the diagram (3.14): by the projection formula, we have

$$Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \overset{L}{\underset{\mathscr{O}_\Delta}{\otimes}} \mathbb{C}(0) = Rf_*\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y) \overset{L}{\underset{f^{-1}\mathscr{O}_\Delta}{\otimes}} f^{-1}\mathbb{C}(0)\right) = Rf_*\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y\right).$$

Our main result concerning the relative log de Rham complex is the following.

Theorem 3.2. *The higher direct image $R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is locally free for each $\ell \in \mathbb{Z}$. Moreover, there exists a canonical isomorphism for every $p \in \Delta$*

$$R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \otimes \mathbb{C}(p) \simeq H^\ell(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_{X_p}), \quad \text{where } \mathbb{C}(p) \text{ is the residue filed at } p.$$

We first present two preliminary theorems.

Theorem 3.3. *The operator R_ℓ has eigenvalues in $[0,1) \cap \mathbb{Q}$ for each $\ell \in \mathbb{Z}$.*

Proof. Later in §4 (Theorem 4.19) we will show that in fact $[\nabla]$ satisfies $p([\nabla]) = 0$ for

$$p(\lambda) = \prod_{i\in I}\prod_{j=0}^{e_i-1}(\lambda - \frac{j}{e_i}).$$

Hence so is $R^\ell f_*[\nabla]$ and this implies the eigenvalues are in $[0,1)\cap\mathbb{Q}$.

Alternatively, by Grothendieck spectral sequence

$$E_2^{p,q} = R^p f_*\mathscr{H}^q(\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \Rightarrow R^{p+q}f_*(\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y),$$

it suffices to show that the induced operator $R^p f_*\mathscr{H}^q[\nabla]$ on $R^p f_*\mathscr{H}^q\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ has eigenvalues in $[0,1)\cap\mathbb{Q}$ for each $q \in \mathbb{Z}$ since $E_\infty^{p,q}$ is a sub-quotient of $E_2^{p,q}$. The following is proved by Steenbrink [Ste76, Proposition 1.13]:

Lemma 3.4. *The stalk of $\mathscr{H}^q\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ at a point u is generated by the germs $(t^{\frac{a}{e}}\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\xi_{i_{q+n}})_u$ for all $0 \le a < e$ and all $0 \le i_1, i_2, ..., i_{q+n} \le n$ over the ring $\mathbb{C}\{t^{\frac{1}{e}}\}/t\mathbb{C}\{t^{\frac{1}{e}}\}$ where e is the* gcd *of $e_0, e_1, ..., e_k$ and $\mathbb{C}\{t^{\frac{1}{e}}\}$ is the ring of convergent power series with the variable $t^{\frac{1}{e}}$.*

We will elaborate the proof of the lemma later. Temporarily admitting the lemma, then

$$\mathscr{H}^q[\nabla]_u(t^{\frac{a}{e}}\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\xi_{i_{q+n}})_u = (\frac{a}{e}t^{\frac{a}{e}}\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\wedge\xi_{i_{q+n}})_u,$$

meaning that the eigenvalues of $\mathscr{H}^q[\nabla]$ are $0, \frac{1}{e}, \frac{2}{e}, ..., \frac{e-1}{e} \in [0,1)\cap\mathbb{Q}$ in a neighborhood of u. This implies that there exists an open neighborhood U containing u and a polynomial $p_U(\lambda)$ whose roots are in $[0,1)\cap\mathbb{Q}$ such that $p_U(\mathscr{H}^q[\nabla]) = 0$ over U. By the properness of Y, we can take a finite open covering $\mathcal{U} = \{U_i\}$ of Y such that $p(\mathscr{H}^q[\nabla]) = \prod_i p_{U_i}(\mathscr{H}^q[\nabla]) = 0$. It follows that $p(R^p f_*\mathscr{H}^q[\nabla]) = 0$, meaning eigenvalues of $R^p f_*\mathscr{H}^q[\nabla]$ in $[0,1)\cap\mathbb{Q}$. ☐

Proof of Lemma 3.4. We will actually prove the original statement of [Ste76, Proposition 1.13] that, in the same notations as in the lemma, the stalk at a point u of $\mathscr{H}^q\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is generated by germs

$$\left(t^{\frac{a}{e}}\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\xi_{i_{q+n}}\right)_u$$

for all $a \in \mathbb{Z}_{\ge 0}$ and all tuples $0 \le i_1, i_2, ..., i_{q+n} \le n$ over $\mathbb{C}\{t^{\frac{1}{e}}\}$ The lemma is a direct corollary.

The complex of stalks $\Omega^{\bullet+n}_{X/\Delta}(\log Y)_u$ can be identified with the Kozul complex of operators $D_1, D_2, ..., D_n$ on $\mathscr{O}_{X,u}$ putting in degree $-n, -n+1, ..., 0$. Define $G^j\Omega^{\ell}_{X/\Delta}(\log Y)_u$ to be the submodules of $\Omega^{\ell}_{X/\Delta}(\log Y)_u$ spanned by the germs

$$\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_\ell} \qquad \text{such that } \#\{m : i_m \le k\} \ge j.$$

Then $\{G^{\ell}\Omega^{\bullet}_{X/\Delta}(\log Y)_u\}_{\ell\in\mathbb{Z}}$ is a decreasing filtration of $\Omega^{\bullet}_{X/\Delta}(\log Y)_u$. The associated spectral sequence has $E_0^{r,\bullet} = \mathrm{gr}^r_G \Omega^{r+\bullet}_{X/\Delta}(\log Y)_u$. Notice that $\mathrm{gr}^r_G\Omega^{r+\bullet}_{X/\Delta}(\log Y)_u$ can be identified with direct sums of Koszul complex of operators $D_{k+1}, D_{k+2}, ..., D_n$ on $\mathscr{O}_{X,u}$, so $E_1^{r,\ell} = H^{r+\ell}(\mathrm{gr}^r_G\Omega^{\bullet}_{X/\Delta}(\log Y)) = 0$ for $\ell \ne 0$ and $E_1^{r,0}$ is spanned by germs

$$\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_\ell} \text{ such that } \#\{i_m \le k\} = j$$

over $\mathbb{C}\{z_0, z_1, ..., z_k\}$, thanks to the usual Poincaré lemma. Consequently, the spectral sequence degenerates at E_2 with $E_2^{r,0} = \mathscr{H}^r(\Omega^{\bullet}_{X/\Delta}(\log Y))_u$. Now $E_1^{\bullet,0}$ is the Koszul complex of operators $D_1, D_2, ..., D_k$ on $\mathbb{C}\{z_0, z_1, ..., z_k\}$. Because each D_i for $0 \le i \le k$ is a homogenous differential operator, E_2 can be computed monomial by monomial.

For simplicity let $\xi_{i_1,i_2,...,i_r} = \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_r}$. Now I claim that a cocycle

$$v = \sum_{i_1<i_2,...<i_r} c_{i_1,i_2,...,i_r} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,..,i_r} \in E_1^{r,0}$$

is cohomologous to zero if $A_j := a_j/e_j - a_0/e_0 \ne 0$ for some $1 \le j \le k$. Note that $D_j(z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k}) = A_j z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k}$ for every $1 \le j \le k$. Since v is a cocycle, the coefficients satisfy

$$\sum_{\ell=1}^{r} (-1)^{\ell} c_{i_1,i_2,...,\hat{i_\ell},...,i_{r+1}} A_{i_\ell} = 0. \tag{3.15}$$

Assume that not all A_j's are zero for $1 \le j \le k$ then $A = \sum A_i^2$ is non-zereo. Then the number

$$d_{i_1,i_2,...,i_{r-1}} = \sum_{\alpha=1}^{k} \frac{A_\alpha}{A} c_{\alpha,i_1,i_2,...,i_{r-1}}.$$

is well-defined. Here we extend standardly that $c_{\sigma(i_1),\sigma(i_2),..,\sigma(i_r)} = \mathrm{sign}(\sigma) c_{i_1,i_2,...,i_r}$ for any permutation σ. Then the element

$$\sum_{i_1<i_2<...<i_{r-1}} d_{i_1,i_2,...,i_{r-1}} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,...,i_{r-1}}$$

in $E_1^{r-1,0}$ has coboundary

$$\begin{aligned}
&\sum_{\alpha=1}^{k} \sum_{i_1<...<i_{r-1}} A_\alpha d_{i_1,i_2,...,i_{r-1}} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{\alpha,i_1,i_2,...,i_{r-1}} \\
&= \sum_{i_1<...<i_r} \sum_{\ell=1}^{r} (-1)^{\ell} A_{i_\ell} d_{i_1,i_2,...,\hat{i_\ell},...,i_r} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,...,i_r} \\
&= \sum_{i_1<...<i_r} \sum_{\alpha=1}^{k} \sum_{\ell=1}^{r} (-1)^{\ell} \frac{A_{i_\ell} A_\alpha}{A} c_{\alpha,i_1,i_2,...,\hat{i_\ell},...,i_r} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,...,i_r} \\
\text{applying (3.15)} \quad &= \sum_{i_1<...<i_r} \sum_{\alpha=1}^{k} \frac{A_\alpha^2}{A} c_{,i_1,i_2,...,i_r} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,...,i_r} = v.
\end{aligned}$$

We conclude the claim. Therefore, $E_2^{r,0}$ is generated over $\mathbb{C}$ by $z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,..,i_r}$ with

$$D_i(z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k}) = 0.$$

That is, $z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} = t^{a/e}$ for some a. Hence, we conclude the lemma. □

Theorem 3.5. *Let $\mathcal{F}^\bullet$ be a complex of $\mathscr{O}_\Delta$-modules with coherent cohomologies, equipped with a log connection, i.e an operator*

$$\nabla \in \operatorname{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}(\mathcal{F}^\bullet) \quad \textit{such that } [\nabla, g] = tg'$$

for ant local holomorphic function g where g' is the derivative of g. Assume that the residue R_ℓ of ∇ defined in the beginning of this subsection acting on each cohomology $\mathscr{H}^\ell\left(\mathcal{F}^\bullet \otimes^L \mathbb{C}(0)\right)$ has eigenvalues in $[0,1)$. Then every $\mathscr{H}^\ell(\mathcal{F}^\bullet)$ is locally free.

Proof. By the definition of residue, we have the morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^\bullet \otimes^L \mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1] \\
\downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle (\nabla+1)[1]} \\
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^\bullet \otimes^L \mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1]
\end{array}$$

in $\mathbf{D}^b(\Delta,\mathbb{C})$. Taking cohomologies gives

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathscr{H}^{\ell-1}\left(\mathcal{F}^\bullet \otimes^L \mathbb{C}(0)\right) & \longrightarrow & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \xrightarrow{t} & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \longrightarrow & \mathscr{H}^\ell\left(\mathcal{F}^\bullet \otimes^L \mathbb{C}(0)\right) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle R_\ell} & & \downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle R_{\ell+1}} & \\
\cdots \longrightarrow & \mathscr{H}^{\ell-1}\left(\mathcal{F}^\bullet \otimes^L \mathbb{C}(0)\right) & \longrightarrow & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \xrightarrow{t} & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \longrightarrow & \mathscr{H}^\ell\left(\mathcal{F}^\bullet \otimes^L \mathbb{C}(0)\right) & \longrightarrow \cdots
\end{array} \tag{3.16}$$

For simplicity, fix ℓ and let $\mathscr{H} = \mathscr{H}^\ell(\mathcal{F}^\bullet)$ and denote by $\ker t$ the kernel of the morphism $t : \mathscr{H} \to \mathscr{H}$. It suffices to prove that $\ker t$ is trivial on $\mathscr{H}$. We are going to show that $\ker t$ is a subset of $t^k\mathscr{H}$ for all $k \geq 0$ and thus, by Krull's theorem $\ker t$ is zero.

It follows from the diagram (3.16) that $\nabla + 1$ on $\ker t$ and ∇ on $\mathscr{H}/t\mathscr{H}$ have eigenvalues in $[0,1)$. Therefore, there exists a polynomial $b_1(s) \in \mathbb{C}[s]$ with roots in $[0,1)$ such that

$$b_1(\nabla)\mathscr{H} \subset t\mathscr{H},$$

and another a polynomial $b_2(s) \in \mathbb{C}[s]$ with eigenvalues in $[0,1)$ such that

$$b_2(\nabla + 1)\ker t = 0.$$

Suppoe v is an element in $\ker t \cap t^k\mathscr{H}$ for some $k \geq 0$. It follows that $v = t^k v_1$ for some $v_1 \in \mathscr{H}$. Because the roots of $b_1(s-k)$ are bigger then the roots of $b_2(s+1)$, the two polynomials $b_1(s-k)$ and $b_2(s+1)$ are relative prime. We deduce that there exist $p(s), q(s) \in \mathbb{C}[s]$ such that

$$1 = p(s)b_1(s-k) + q(s)b_2(s+1).$$

Therefore, combining the fact that $b_2(\nabla+1)v$ vanishes,

$$v = p(\nabla)b_1(\nabla - k)v + q(\nabla)b_2(\nabla+1)v = p(\nabla)b_1(\nabla-k)t^k v_1.$$

Because of the identity $(\nabla - k)t^k = t^k\nabla$, the above is equivalent to

$$v = t^k p(\nabla + k)b_1(\nabla)v_1.$$

Because $b_1(\nabla)v_1 = tv_2$ for some $v_2 \in \mathscr{H}$, substituting in the last equality yields

$$v = t^k p(\nabla+k)b_1(\nabla)v_1 = t^k p(\nabla+k)b_1(\nabla)tv_2 = t^{k+1}p(\nabla+k+1)b_1(\nabla+1)v_2 \in t^{k+1}\mathscr{H}.$$

We proved that v is also an element in $t^{k+1}\mathscr{H}$. By induction and Krull's theorem we conclude the proof. ∎

Now we can immediately finish

Proof of Theorem 3.2. The complex $Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ with $Rf_*\nabla$ satisfies the condition of Theorem 3.5. Therefore, each cohomology $R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is locally free. The second statement in the theorem follows from the the locally freeness of $R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ plus the Grauert's base change theorem. □

4. Transfer to $\mathscr{D}$–modules

Lemma 3.4 implies the restriction of the relative log de Rham complex on Y is semi-perverse. Indeed, it is even perverse, showed in [Ste76, §2]. Therefore, there should be a regular holonomic $\mathscr{D}$-module whose de Rham complex is the restriction of the relative log de Rham complex on Y, in the view of Riemann-Hilbert correspondence established by Kashiwara [Kas84] and Mebkhout [Meb84]. The stupid filtration should also translates to a coherent filtration from Hodge theoretic point of view. Then the endomorphism $[\nabla]$ in the derived category can be captured by an endomorphism of a $\mathscr{D}$-module. This enable us to study the relation between the filtration and $[\nabla]$ much easier and cleaner. In this section, we will construct the filtered $\mathscr{D}$-module and the endomorphism.

4.1. **Construction of filtered holonomic $\mathscr{D}_X$-modules.** Since $\mathscr{T}_{X/\Delta}(\log)$ is a subsheaf of $\mathscr{T}_X$, the multiplication by sections in $\mathscr{T}_{X/\Delta}(\log Y)$ induces a morphism $\mathscr{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$, with $P \mapsto \sum_{i=1}^k \xi_i \otimes D_i P$ locally. The morphism extends to a filtered complex of $\mathscr{D}_X$-modules

$$\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X = \{\mathscr{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes \mathscr{D}_X \to \cdots \to \Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\}[n] \tag{4.17}$$

with filtration $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$ given by

$$\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes F_{\ell+n+\bullet}\mathscr{D}_X = \{F_\ell\mathscr{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes F_{\ell+1}\mathscr{D}_X \to \cdots \to \Omega^n_{X/\Delta}(\log Y) \otimes F_{\ell+n}\mathscr{D}_X\}[n].$$

Let $\tilde{\mathcal{M}}$ be the 0-th cohomology of $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$ and $F_\ell\mathcal{M}$ be the $\mathscr{O}_X$-submodule induced by the the filtration $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$.

Theorem 4.1. *The complex $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$ is a filtered resolution of a filtered $\mathscr{D}_X$-module $(\tilde{\mathcal{M}}, F_\bullet\tilde{\mathcal{M}})$.*

Proof. Notice that $\mathrm{gr}^F\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right) = \Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathrm{gr}^F\mathscr{D}_X$, can be identified locally with the Koszul complex associated to the regular sequence $D_1, D_2, \dots, D_n$ over the ring $\mathrm{gr}^F\mathscr{D}_X$. It follows that $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathrm{gr}^F\mathscr{D}_X$ is acyclic. Therefore, each graded peace $\mathrm{gr}^F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$ is acyclic. We deduce inductively that $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$ is also acyclic; this can be seen from the long exact sequence associated to the short exact sequence

$$0 \to F_{\ell-1}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right) \to F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right) \to \mathrm{gr}^F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right) \to 0.$$

Taking direct limit, we conclude that $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$ is a resolution of $\tilde{\mathcal{M}}$. The long exact sequence also implies the 0-th cohomology of $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$ is isomorphic to $F_\ell\tilde{\mathcal{M}}$. This completes the proof. □

Remark 4.2. Note that $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$ is a complex of $(f^{-1}\mathscr{O}_\Delta, \mathscr{D}_X)$-bimodules because $\Omega^{n+\bullet}_{X/\Delta}(\log Y)$ is $f^{-1}\mathscr{O}_\Delta$-linear. It follows that $\tilde{\mathcal{M}}$ is also a $(f^{-1}\mathscr{O}_\Delta, \mathscr{D}_X)$-bimodule. Note we have two different actions of t on $\tilde{\mathcal{M}}$ due to the bimodule structure. We usually use the left multiplication by t. One can think of $\tilde{\mathcal{M}}$ as a flat family assembling the $\mathscr{D}$-module $i_{X_p +}\omega_{X_p}$ of the smooth fibers X_p for $p \in \Delta$ and a specialization $\mathcal{M} = \tilde{\mathcal{M}}/t\tilde{\mathcal{M}}$ because using the left $f^{-1}\mathscr{O}_\Delta$ structure, we have filtered isomorphisms

$$\mathbb{C}(p) \otimes \tilde{\mathcal{M}} \simeq \mathbb{C}(p) \otimes \Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X \simeq \Omega^{n+\bullet}_{X/\Delta}(\log Y)|_{X_p} \otimes \mathscr{D}_X \simeq i_{X_p *}\Omega^{n+\bullet}_{X_p} \otimes \mathscr{D}_X \simeq i_{X_p +}\omega_{X_p},$$

where $i_{X_p} : X_p \to X$ is the closed embedding of the smooth fiber X_p.

Remark 4.3. The theorem also says by choosing the local trivialization $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$ of $\Omega^n_{X/\Delta}(\log Y)$, $\tilde{\mathcal{M}}$ can be identified locally with $\mathscr{D}_X/(D_1, D_2, ..., D_n)\mathscr{D}_X$ and $\mathrm{gr}^F \tilde{\mathcal{M}}$ can be identified locally with $\mathrm{gr}^F \mathscr{D}_X/(D_1, D_2, ..., D_n)\mathrm{gr}^F \mathscr{D}_X$.

Remark 4.4. Let $\mathscr{D}_{X/\Delta}(\log Y)$ be the subalgebra of $\mathscr{D}_X$ generated by $\mathscr{T}_{X/\Delta}(\log Y)$. One can show that $\tilde{\mathcal{M}}$ is nothing but

$$\omega_{X/\Delta}(\log Y) \underset{\mathscr{D}_{X/\Delta}(\log Y)}{\otimes} \mathscr{D}_X.$$

And the filtration $F_\bullet \tilde{\mathcal{M}}$ is induced from $F_\bullet \omega_{X/\Delta}(\log Y)$, where $F_\ell \omega_{X/\Delta}(\log Y)$ is $\omega_{X/\Delta}(\log Y)$ for $\ell \geq -n$ and is zero otherwise. To keep the proof elementary, we avoid talking about $\mathscr{D}_{X/\Delta}(\log Y)$-modules.

Theorem 4.5. *The complex $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y \otimes \mathscr{D}_X$ is a filtered resolution of a filtered holonimic $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet \mathcal{M})$.*

Proof. Because of the bimodule structure, we have $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y \otimes \mathscr{D}_X$ is the cokernel of the left multiplication by t on $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$. Therefore, the first part of the statement is equivalent to $t : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is injective. It suffices to prove that $t : \mathrm{gr}^F \tilde{\mathcal{M}} \to \mathrm{gr}^F \tilde{\mathcal{M}}$ is injective because the multiplication by t is a filtered morphism. But this follows from $t, D_1, D_2, ..., D_n$ is a regular sequence over the ring $\mathrm{gr}^F \mathscr{D}_X$. It also follows that $\mathrm{gr}^F \mathcal{M}$ is isomorphic locally to $\mathrm{gr}^F \mathscr{D}_X/(t, D_1, D_2, ..., D_n)\mathrm{gr}^F \mathscr{D}_X$. This means the characteristic variety of $\mathcal{M}$ is cut out by $t, D_1, D_2, ..., D_n \in \mathscr{O}_{T^*X}$ and thus, the characteristic variety is of dimension $n+1$. This proves the holonomicity of $\mathcal{M}$. ☐

Remark 4.6. Similarly to the case of $\tilde{\mathcal{M}}$, the $\mathscr{D}_X$-module $\mathcal{M}$ is just

$$\omega_{X/\Delta}(\log Y)|_Y \underset{\mathscr{D}_{X/\Delta}(\log Y)}{\otimes} \mathscr{D}_X$$

with the filtration $F_\bullet \mathcal{M}$ induced by $\left(F_\bullet \omega_{X/\Delta}(\log Y)\right)|_Y$.

4.2. **Properties of $\mathcal{M}$.** We first calculate the characteristic cycle of $\mathcal{M}$ which is important for later when we identifying the primitive part of $\mathrm{gr}^W \mathcal{M}$. Then we prove that the de Rham complex of $\mathcal{M}$ with the induced filtration recover $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ with the stupid filtration. Lastly, we translate the operator $[\nabla] \in \mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}(\Omega^{\bullet+n}_{X/\Delta}(\log Y))|_Y$ to an operator R on $\mathcal{M}$

Theorem 4.7. *The characteristic cycle of $\mathcal{M}$ is*

$$cc(\mathcal{M}) = \sum_{J \subset I} \sum_{j \in J} e_j \, [T^*_{Y^J} X],$$

where $[T^*_{Y^J} X]$ is the cycle of the conormal bundle of Y^J in T^*X and e_i is the multiplicity of Y along each component Y_i for $i \in I$.

Proof. The statement is local and we identify $\mathcal{M}$ with $\mathscr{D}_X/(t, D_1, D_2, ..., D_n)$. We first describe the characteristic variety of $\mathcal{M}$. The support of $\mathrm{gr}^F \mathcal{M}$ as a sheaf on T^*X is defined by the radical of the ideal $(t, D_1, D_2, ..., D_n)\mathrm{gr}^F \mathscr{D}_X$. In fact, $z_i \partial_i$ for $0 \leq i \leq k$ is in the radical because

$$(z_i \partial_i)^{e_0 + e_1 + \cdots + e_k} \equiv (z_0 \partial_0)^{e_0} (z_1 \partial_1)^{e_1} \cdots (z_k \partial_k)^{e_k} \equiv t \partial_0^{e_0} \partial_1^{e_1} \cdots \partial_k^{e_k} \equiv 0 \bmod (t, D_1, D_2, ..., D_n)\mathrm{gr}^F \mathscr{D}_X.$$

Therefore, $char(\mathcal{M})$ is cut out by $t_{\mathrm{red}}, z_0 \partial_0, z_1 \partial_1, ..., z_k \partial_k, \partial_{k+1}, ..., \partial_n$, where $t_{\mathrm{red}} = z_0 z_1 \cdots z_k$. It follows that $char(\mathcal{M}) = \bigcup_{J \subset I} T^*_{Y^J} X$.

Denote by $\mathfrak{p}(Z)$ the prime ideal defining a integral subvariety Z. Let m_J be the length of $\mathrm{gr}^F \mathcal{M}_{\mathfrak{p}(T^*_{Y^J} X)}$ as an Artinian $\mathrm{gr}^F \mathscr{D}_{X,\mathfrak{p}}$-module . Then $cc(\mathcal{M}) = \sum_{J \in I} m_J \left[T^*_{Y^J} X\right]$. For simplicity let us assume $J = \{0, 1, 2, .., \mu\}$

and by abuse of notation we also the prime ideal $\mathfrak{p} = \mathfrak{p}(T^*_{Y^J}X)$ of the variety $T^*_{Y^J}X$ is locally generated by $z_0, z_1, ..., z_\mu, \partial_{\mu+1}, \partial_{\mu+2}, ..., \partial_n$ over $\mathrm{gr}^F\mathscr{D}_X$ in some local coordinate system. Notice that

$$\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}/(t, D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}} = \mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}/(D'_0, D'_1, ..., D'_n)\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}$$

where

$$D'_i = \begin{cases} z_0^{e_0+e_1+\cdots+e_\mu}, & \text{for } i = 0 \\ \dfrac{1}{e_i}z_i - \dfrac{1}{e_0}z_0\dfrac{\partial_0}{\partial_i}, & \text{for } 1 \le i \le \mu \\ \dfrac{1}{e_i}\partial_i - \dfrac{1}{e_0}z_0\dfrac{\partial_0}{z_i}, & \text{for } \mu+1 \le i \le k \\ \partial_i, & \text{for } i > k, \end{cases} \tag{4.18}$$

because $\partial_0, \partial_1, ..., \partial_\mu, z_{\mu+1}, z_{\mu+2}, ..., z_k$ are invertible in $\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}$. Therefore, $\mathrm{gr}^F\mathcal{M}_\mathfrak{p}$ can be identifies with

$$\mathbb{C}\{z_0\}/(z_0^{e_0+e_1+\cdots+e_\mu}).$$

Then $m_J = \dim_\mathbb{C}\mathbb{C}\{z_0\}/(z_0^{e_0+e_1+\cdots+e_\mu}) = \sum_{j\in J} e_j$. This completes the computation. ☐

Remark 4.8. The above theorem verifies that $cc(\mathcal{M}) = \lim_{p\to 0} cc(i_{p_+}\omega_{X_p}) = \lim_{p\to 0}\left[T^*_{X_p}X\right]$ as cycles in algebraic cotangent space T^*X for $p \in \Delta^*$ where $i_p : X_p \to X$ the closed embedding of the smooth fiber. In fact, one can show that $\mathbb{C}(p)\otimes\mathrm{gr}^F\tilde{\mathcal{M}}$, using the left $f^{-1}\mathscr{O}_\Delta$-module structure, is isomorphic to $\mathrm{gr}^F i_{p_+}\omega_{X_p}$ as in Remark 4.2. Refer to [Gin86] for general results about the characteristic cycles of specializations of holonomic $\mathscr{D}$-modules.

Corollary 4.9. *The de Rham complex $\mathrm{DR}_X\mathcal{M}$ together with filtration $F_\bullet\mathrm{DR}_X\mathcal{M}$ is isomorphic to $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y$ with the stupid filtration in the derived category of filtered complexes of sheaves of $\mathbb{C}$-vector spaces.*

Proof. We have showed that $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)$ is a resolution of $F_\ell\mathcal{M}$. Therefore, the total complex of $F_{\ell+*}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\otimes\bigwedge^{-*}\mathscr{T}_X$ is quasi-isomorphic to $F_{\ell+*}\mathcal{M}\otimes\bigwedge^{-*}\mathscr{T}_X$, which is exactly $F_\ell\mathrm{DR}_X\mathcal{M}$. It remains to show the total complex also quasi-isomorphic to $F_\ell\Omega^{n+\bullet}_{X/\Delta}(\log Y)$. This follows from that

$$\begin{aligned} F_{\ell+*}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\otimes\bigwedge^{-*}\mathscr{T}_X &= \Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes F_{\ell+n+\bullet}\left(\mathscr{D}_X\otimes\bigwedge^{-*}\mathscr{T}_X\right) \\ &\simeq \Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes F_{\ell+n+\bullet}\mathscr{O}_X \\ &= F_\ell\Omega^{n+\bullet}_{X/\Delta}(\log Y). \end{aligned}$$

Here, $F_\ell\mathscr{O}_X = \mathscr{O}_X$ for $\ell \ge 0$ and otherwise it is zero. ☐

Theorem 4.10. *The endomorphism $\nabla \in \mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}\Omega^{n+\bullet}_{X/\Delta}(\log Y)$ in Lemma 3.1 transfers to a filtered morphism*

$$\nabla : (\tilde{\mathcal{M}}, F_\bullet\tilde{\mathcal{M}}) \to (\tilde{\mathcal{M}}, F_{\bullet+1}\tilde{\mathcal{M}}), \quad [[\alpha]\otimes P] \mapsto (\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}$$

where $\alpha \in \Omega^n_X(\log Y)$ and $P \in \mathscr{D}_X$ so that $[\alpha]\otimes P \in \Omega^n_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$. Moreover, restriction on Y yields a filtered morphism

$$R : (\mathcal{M}, F_\bullet\mathcal{M}) \to (\mathcal{M}, F_{\bullet+1}\mathcal{M})$$

such that

$$\prod_{i\in I}\prod_{j=0}^{e_i-1}(R - \frac{j}{e_i}) = 0. \tag{4.19}$$

Proof. The morphism $\frac{dt}{t}\wedge : \Omega^{n+\bullet}_{X/\Delta}(\log Y) \to \Omega^{n+1+\bullet}_X(\log Y)$ extends to the corresponding complexes of induced $\mathscr{D}_X$-modules

$$\frac{dt}{t}\wedge : \Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes \mathscr{D}_X \to \Omega^{n+1+\bullet}_X(\log Y)\otimes \mathscr{D}_X.$$

Let $\mathrm{Cone}^\bullet \otimes \mathscr{D}_X$ be the mapping cone of the above morphism. We get a diagram of complexes of $\mathscr{D}_X$-modules similarly to (3.13) and taking 0-th cohomology we get the following.

$$\begin{array}{ccc} \mathscr{H}^0\left(\mathrm{Cone}^\bullet \otimes \mathscr{D}_X\right) & \xrightarrow{\ q\ } & \tilde{\mathcal{M}} \\ \downarrow p & \swarrow {\scriptstyle p\circ q^{-1}} & \\ \tilde{\mathcal{M}} & & \end{array} \tag{4.20}$$

where abuse of notation, still denote by p and q the induced morphisms from diagram (3.13). Now q is an isomorphism of $\mathscr{D}_X$-modules. Let $[\alpha\otimes P, [\beta]\otimes Q]$ be a class in $\mathscr{H}^0(\mathrm{Cone}^\bullet \otimes \mathscr{D}_X)$ for any $\alpha\otimes P \in \Omega^n_X(\log Y)\otimes\mathscr{D}_X$ and $[\beta]\otimes Q \in \Omega^n_{X/\Delta}(\log Y)\otimes \mathscr{D}_X$. Then

$$\delta\left(\alpha\otimes P, [\beta]\otimes Q\right) = \left((-1)^n d(\alpha\otimes P) + \frac{dt}{t}\wedge\beta\otimes Q, (-1)^n d([\beta]\otimes Q)\right) = 0.$$

Here, the sign factor $(-1)^n$ shows up due to we follow the Koszul sign rule. Because $\frac{dt}{t}\wedge : \Omega^n_{X/\Delta}(\log Y) \to \Omega^{n+1}_X(\log Y)$ is an isomorphism, we have

$$[\beta]\otimes Q = (-1)^{n-1}(\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}.$$

Therefore, $q^{-1} : \tilde{\mathcal{M}} \to \mathscr{H}^0(\mathrm{Cone}^\bullet\otimes\mathscr{D}_X)$ is given by $[[\alpha]\otimes P] \mapsto [\alpha\otimes P, (-1)^n(\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}]$. Then we have

$$\nabla = (-1)^{n-1} p\circ q^{-1} : [[\alpha]\otimes P] \mapsto (\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}.$$

Restricting to Y we have the induced operator R on $\mathcal{M}$. If $\alpha = \xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n$ then

$$\begin{aligned} R[\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n\otimes P] &= (\frac{dt}{t}\wedge)^{-1}\left(d\left(e_1\frac{dz_1}{z_1}\wedge e_2\frac{dz_2}{z_2}\wedge\cdots\wedge dz_n\otimes P\right)\right) \\ &= (\frac{dt}{t}\wedge)^{-1}\left(e_0\frac{dz_0}{z_0}\wedge e_1\frac{dz_1}{z_1}\wedge e_2\frac{dz_2}{z_2}\wedge\cdots\wedge dz_n\otimes \frac{1}{e_0}z_0\partial_0 P\right) \\ &= \left[\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n\otimes\frac{1}{e_0}z_0\partial_0 P\right]. \end{aligned}$$

We see that $RF_\bullet\mathcal{M} \subset F_{\bullet+1}\mathcal{M}$. The reason for $\nabla F_\bullet\tilde{\mathcal{M}} \subset F_{\bullet+1}\tilde{\mathcal{M}}$ is similar. To prove the last statement, we work locally and identify $\mathcal{M}$ with $\mathscr{D}_X/(t, D_1, ..., D_n)$ via the local trivialization $\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n$ of $\Omega^n_{X/\Delta}(\log Y)$. Then for $P\in\mathscr{D}_X$, $R[P] = [\frac{1}{e_0}z_0\partial_0 P]$. In fact, because of the relation $D_1, D_2, ..., D_n$, the left multiplication by $\frac{1}{e_0}z_0\partial_0$ on $\mathcal{M}$ is the same as the multiplication by $\frac{1}{e_i}z_i\partial_i$ for $1\le i\le k$. It follows from the identity

$$(z\partial)(z\partial-1)\cdots(z\partial-\ell) = z^{\ell+1}\partial^{\ell+1}$$

for any $\ell\ge 0$ that

$$\begin{aligned} \prod_{i\in I}\prod_{j=0}^{e_i-1}(R-\frac{j}{e_i})[P] &= \prod_{i\in I}\prod_{j=0}^{e_i-1}(\frac{1}{e_i}z_i\partial_i - \frac{j}{e_i})[P] = \prod_{i\in I}\frac{1}{e_i^{e_i}}z_i^{e_i}\partial_i^{e_i}[P] = t\prod_{i\in I}\frac{1}{e_i^{e_i}}\partial_i^{e_i}[P] \\ &= 0 \in \mathscr{D}_X/(D_1, D_2, ..., D_n, t)\mathscr{D}_X. \end{aligned}$$

This completes the proof. $\square$

Remark 4.11. Note that $\nabla : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is also can be identified with the left multiplication by $\frac{1}{e_i} z_i \partial_i$ for $i \le k$, by choosing the trivialization of $\Omega^n_{X/\Delta}(\log Y)$, because of the relations $D_i = \frac{1}{e_i} z_i \partial_i - \frac{1}{e_0} z_0 \partial_0$ for $1 \le i \le k$. This means for any function $g \in f^{-1}\mathscr{O}_\Delta$, we have $[\nabla, g] = tg'$ where t and g are local sections of $f^{-1}\mathscr{O}_\Delta$ acting on the left of $\tilde{\mathcal{M}}$. This makes $\tilde{\mathcal{M}}$ a $(f^{-1}\mathscr{D}_\Delta(\log 0), \mathscr{D}_X)$-bimodule. Using Godement resolution, the direct image $Rf_*\mathrm{DR}_X\tilde{\mathcal{M}}$ is a complex of left $\mathscr{D}_\Delta(\log 0)$-modules. Similarly, as we already saw in the proof, locally the morphism $R : \mathcal{M} \to \mathcal{M}$ can be identified with left multiplication by $\frac{1}{e_i} z_i \partial_i$ for $0 \le i \le k$, meaning $[R, g] = tg' = 0$ for g local sections of $f^{-1}\mathscr{O}_\Delta$ acting left on $\mathcal{M}$.

Remark 4.12. The $\mathscr{D}_X$-module $\mathcal{M}$ is even regular holonomic. Even though it is irrelevant for our purpose, we can also check $\mathcal{M}$ is regular using the definition. Recall that a holonomic right $\mathscr{D}_Z$-module $\mathcal{N}$ is called *regular* if there exists a good filtration $F_\bullet\mathcal{N}$ such that for any $\sigma \in \mathrm{gr}^F\mathscr{D}_Z$ vanishing on the charateristic variety of $\mathcal{N}$ one has $\mathrm{gr}^F\mathcal{N}\sigma = 0$. In the case of $\mathcal{M}$, define locally

$$G_\ell\mathcal{M} = \sum_{r,k\ge 0} R^k F_{\ell+r}\mathcal{M} t^r_{\mathrm{red}}$$

where $t_{\mathrm{red}} = z_0 z_1 \cdots z_k$. This is a finite sum because $\mathcal{M}$ is supported on $t = 0$ and R has a characteristic polynomial. It follows that $G_\bullet$ is a good filtration for $\mathcal{M}$. I claim that $G_\bullet\mathcal{M}$ gives the filtration in the definition of the regularity. Since the characteristic variety of $\mathcal{M}$ is locally cut out by $t_{\mathrm{red}}, z_0\partial_0, \ldots, z_k\partial_k, \partial_{k+1}, \ldots, \partial_n$ (see Theorem 4.7) it suffices to check that $G_\ell\mathcal{M}t_{\mathrm{red}} \subset G_{\ell-1}\mathcal{M}$, $G_\ell\mathcal{M}z_i\partial_i \subset G_\ell\mathcal{M}$ for $0 \le i \le k$ and $G_\ell\mathcal{M}\partial_i \subset G_\ell\mathcal{M}$ for $k+1 \le i \le n$. It is clear that $G_\ell\mathcal{M}t_{\mathrm{red}} \subset G_{\ell-1}\mathcal{M}$. Due to locally $\mathrm{gr}^F\mathcal{M} = \mathrm{gr}^F\mathscr{D}_X/(t, D_1, D_2, \ldots, D_n)\mathrm{gr}^F\mathcal{M}$, it follows that $\mathrm{gr}^F\mathcal{M}D_i = 0$ for $1 \le i \le n$. In particular, $\mathrm{gr}^F\mathcal{M}\partial_i = 0$ for $k+1 \le i \le n$, i.e. $F_\ell\mathcal{M}\partial_i \subset F_\ell\mathcal{M}$ for $k+1 \le i \le n$. Therefore, for $k+1 \le i \le n$, because $[t_{\mathrm{red}}, \partial_i] = 0$,

$$G_\ell\mathcal{M}\partial_i = \sum_{r,k\ge 0} R^k F_{\ell+r}\mathcal{M}t^r_{\mathrm{red}}\partial_i \subset \sum_{r,k\ge 0} R^k F_{\ell+r}\mathcal{M}t^r_{\mathrm{red}} = G_\ell\mathcal{M}.$$

Since $[t^r_{\mathrm{red}}, z_i\partial_i] = (z_i\partial_i - r)\, t^r_{\mathrm{red}}$, and $[z_i\partial_i, F_\ell\mathscr{D}_X] \subset F_\ell\mathscr{D}_X$, we have

$$R^k F_{\ell+r}\mathcal{M}t^r_{\mathrm{red}} z_i\partial_i = R^k F_{\ell+r}\mathcal{M}\,(z_i\partial_i - r)\, t^r_{\mathrm{red}} \subset R^k\,(z_i\partial_i F_{\ell+r}\mathcal{M} + F_{\ell+r}\mathcal{M})\, t^r_{\mathrm{red}}.$$

But locally R has the same effect as the left multiplication by one of $\frac{1}{e_i} z_i\partial_i$ for $0 \le i \le k$. Hence,

$$R^k\,(z_i\partial_i F_{\ell+r}\mathcal{M} + F_{\ell+r}\mathcal{M})\, t^r_{\mathrm{red}} = R^{k+1} F_{\ell+r}\mathcal{M}t^r_{\mathrm{red}} + R^k F_{\ell+r}\mathcal{M}t^r_{\mathrm{red}}.$$

It follows that $G_\ell\mathcal{M}z_i\partial_i \subset G_\ell\mathcal{M}$ for $0 \le i \le k$.

In fact, later we will see that $\mathcal{M}$ is an extensions of regular holonomic $\mathscr{D}_X$-modules which will again prove that $\mathcal{M}$ is regular (see Theorem 5.7 for the reduced case and Theorem 7.13 for the general case).

5. Reduced case: Strictness and the weight filtration

We begin to study the weight filtration $W_\bullet\mathcal{M}$ induced R on $\mathcal{M}$. For simplicity to state the results and illustrate the ideas, we assume Y is reduced in §5 and §6. The general case will be treated in §7 and §8. Since Y is reduced, the multiplicity e_i of irreducible component Y_i is 1 and R is nilpotent. Recall that the weight filtration of the nilpotent operator R is uniquely characterized by the following two properties:

- for each $\ell \in \mathbb{Z}$, $R : W_\ell\mathcal{M} \to W_{\ell-2}\mathcal{M}$;
- the induced operator $R^\ell : \mathrm{gr}^W_\ell\mathcal{M} \to \mathrm{gr}^W_{-\ell}\mathcal{M}$ is an isomorphism for each $\ell \ge 0$.

5.1. **Strictness of R.** Let $F_\bullet W_r\mathcal{M} = F_\bullet\mathcal{M} \cap W_r\mathcal{M}$ be the induced filtration for every integer r. In fact, the good filtration and the weight filtration interact nicely because of the following theorem.

Theorem 5.1. *The power of R is strict on $(\mathcal{M}, F_\bullet\mathcal{M})$, i.e., $R^a F_b\mathcal{M} = F_{a+b}R^a\mathcal{M}$.*

Proof. The strictness is a local property; therefore, we can assume $\mathcal{M} = \mathscr{D}_X/(t, D_1, D_2, ..., D_n)\mathscr{D}_X$ and R is left multiplication by $z_0\partial_0$ on it, recalling that $D_i = z_i\partial_i - z_0\partial_0$ for $1 \le i \le k$ and $D_i = \partial_i$ for $k+1 \le i \le n$. It is clear that $R^a F_b\mathcal{M}$ is contained in $F_{a+b}R^a\mathcal{M}$. It suffices to show that for every $R^a P \in F_{a+b}\mathcal{M}$, we can find an element $Q \in F_b\mathcal{M}$ such that $R^a P = R^a Q$. Assume $P \in F_\ell\mathcal{M}$. If $\ell \le b$ then there is nothing to prove. Thus, we consider the situation that $\ell > b$. Then the class of $R^a P$ vanishes in $\mathrm{gr}^F_{a+\ell}\mathcal{M}$. In fact, we have the following lemma:

Lemma 5.2. *Denote by $[R]$ the induced operator on $\mathrm{gr}^F\mathcal{M}$. Then $\ker[R]^{r+1}$ is locally generated by the classes of all degree $k-r$ monomials dividing $t = z_0 z_1 \cdots z_k$.*

We can easily check that monomials of degree $k-r$ dividing t is in $\ker[R]^{r+1}$. Indeed, it is already true that monomials of degree $k-r$ dividing t is in $\ker R^{r+1}$. Without loss of generality, we only need to check this for the monomial $z_{r+1}z_{r+2}\cdots z_k$:

$$R^{r+1}z_{r+1}z_{r+2}\cdots z_k = z_0\partial_0 z_1\partial_1\cdots z_r\partial_r z_{r+1}z_{r+2}\cdots z_k = t\partial_0\cdots\partial_k = 0 \in \mathcal{M}.$$

We will prove the opposite direction after finishing the proof of the theorem. Going back to the proof of the theorem, by the above lemma,

$$P = \sum_{\substack{J\subset I,\\ \#J=k-a+1}} z_J Q_J + Q_{\ell-1}$$

where $z_J = \prod_{j\in J} z_j$, $Q_J \in F_\ell\mathcal{M}$ and $Q_{\ell-1} \in F_{\ell-1}\mathcal{M}$. But R^a kills the monomials z_J of degree $k-a+1$ dividing t. It follows that $R^a P = R^a Q_{\ell-1}$. Iterating the procedure, we eventually find an element $Q \in F_b\mathcal{M}$ such that $R^a P = R^a Q$ with $Q \in F_b\mathcal{M}$. ☐

Proof of Lemma 5.2. Note that we are over the commutative ring $\mathrm{gr}^F\mathscr{D}_X$. We proceed by induction on r. Let $P \in \mathrm{gr}^F\mathscr{D}_X$ be a representative of an element in $\ker[R]^{r+1}$. When $r = 0$, we have

$$z_0\partial_0 P = tQ_0 + \sum_{i=1}^n D_i Q_i.$$

Then $tQ_0 \in (\partial_0, \partial_1, ..., \partial_n)\mathrm{gr}^F\mathscr{D}_X$. Notice that $t, \partial_0, \partial_1, ..., \partial_n$ is a regular sequence over $\mathrm{gr}^F\mathscr{D}_X$. We have $Q_0 = \sum_{i=0}^n \partial_i Q_i'$. This implies

$$\begin{aligned} z_0\partial_0 P &= \sum_{i=0}^k \frac{t}{z_i} z_i\partial_i Q_i' + \sum_{j=k+1}^n t\partial_j Q_j' + \sum_{i=1}^n D_i Q_i \\ &= \sum_{i=0}^k \frac{t}{z_i} z_0\partial_0 Q_i' + \sum_{i=1}^k D_i(Q_i + \frac{t}{z_i}Q_i') + \sum_{j=k+1}^n D_j(Q_j + tQ_j'), \end{aligned}$$

from which we conclude that $z_0\partial_0(P - \sum_{i=0}^k \frac{t}{z_i}Q_i') \in (D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X$. Because $z_0\partial_0, D_1, D_2, ..., D_n$ is again a regular sequence, we see that $P - \sum_{i=0}^k \frac{t}{z_i}Q_i' \in (D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X$. This concludes the base case for the induction.

Assume the statement is true for the cases when the exponent is less then $r+1$. Let $z_J = \prod_{j\in J} z_j$. Now for $[P] \in \ker[R]^{r+1}$, we have $[R][P]$ is in $\ker[R]^r$. By induction,

$$z_0\partial_0 P = \sum_{\substack{\#J=k-r+1,\\ J\subset I}} z_J Q_J + \sum_{i=1}^n D_i Q_i. \tag{5.21}$$

Fix an index subset J of I such that $\#J = k-r+1$. Then $z_J Q_J$ is in the submodule generated by z_i for $i \in I \smallsetminus J$ and ∂_j for $j \in J$ and $k < j \le n$ over $\mathrm{gr}^F \mathscr{D}_X$. Since z_i for $i \in I \smallsetminus J$, ∂_j for $j \in J$ and $k < j \le n$ and z_J form a regular sequence, we have

$$Q_J = \sum_{i \in I \smallsetminus J} z_i Q_i' + \sum_{j \in J} \partial_j Q_j' + \sum_{k < \ell \le n} \partial_\ell Q_\ell'.$$

Therefore, it follows that

$$z_J Q_J = \sum_{i \in I \smallsetminus J} z_J z_i Q_i' + \sum_{j \in J} \left(\frac{z_J}{z_j} z_0 \partial_0 Q_j' + D_j \frac{z_J}{z_j} Q_j' \right) + \sum_{k < \ell \le n} D_\ell z_J Q_\ell'.$$

Then substuiting in (5.21), we deduce that

$$z_0 \partial_0 \left(P - \sum_{j \in J} \frac{z_J}{z_j} Q_j' \right) - \sum_{i \in I \smallsetminus J} z_J z_i Q_i'$$

is in the submodule generated by degree $k-r+1$ monomials dividing t except z_J, and $D_1, D_2, \ldots, D_n$ over $\mathrm{gr}^F \mathscr{D}_X$. It follows that we can reduce the monomials of degree $k-r+1$ dividing t in the right-hand side equation (5.21) one by one and at the last step, we get $z_0 \partial_0 (P - P') - Q'$, where P' is a linear combination of degree $k-r$ monomials dividing t and Q' is a linear combination of $k-r+2$ monomials dividing t, is in the submodule generated by $D_1, \ldots, D_n$ over $\mathrm{gr}^F \mathscr{D}_X$. But $\ker[R]^{r-1}$ is generated by classes represented by degree $k-r+2$ monomials dividing t by induction hypothesis. It says that the class of $P - P'$ is in $\ker[R]^r$ and by induction it is generated by degree $k-r+1$ monomials dividing t. Therefore, P is a linear combination of degree $k-r$ monomials dividing t. This completes the proof. □

Corollary 5.3. *The* $\ker R^{r+1}$ *is also generated by degree* $k-r$ *monomials dividing* t *if one identifies* $\mathcal{M}$ *locally with* $\mathscr{D}_X / (t, D_1, D_2, \ldots, D_n) \mathscr{D}_X$.

Proof. It suffices to show that $\mathrm{gr}^F \ker R^{r+1}$ is generated by degree $k-r$ monomials dividing t. Notice that $\mathrm{gr}^F \ker R^{r+1}$ is contained in $\ker[R]^{r+1}$, since $[R]^{r+1}$ vanishes on $\mathrm{gr}^F \ker R^{r+1}$. In fact, we have $\mathrm{gr}^F \ker R^{r+1} = \ker[R]^{r+1}$ because degree $k-r$ monomials dividing t are also in $\mathrm{gr}^F \ker R^{r+1}$. □

5.2. **The weight filtration.** The results concerning the weight filtration and Lefschetz decomposition are formal and we will work on the abstract setting.

Theorem 5.4. *Let* $N : (\mathcal{G}, F_\bullet) \to (\mathcal{G}, F_{\bullet+1})$ *be a nilpotent operator on a filtered* $\mathscr{D}$*-module* $(\mathcal{G}, F_\bullet)$*. Asume that every power of* N *satisfies strictness, i.e.,* $N^a F_b \mathcal{G} = F_{a+b} N^a \mathcal{G}$ *for* $a \ge 0$ *and* $b \in \mathbb{Z}$*. Then the induced operator* $N^r : F_\ell \mathrm{gr}_r^W \mathcal{G} \to F_{\ell+r} \mathrm{gr}_{-r}^W \mathcal{G}$ *is an isomorphism for* $r \ge 0$*, where* $W_\bullet$ *is the weight filtration induced by* N*.*

Proof. It suffices to prove that for any $b \in F_{\ell+r} W_{-r} \mathcal{G}$, we could find $a' \in F_\ell W_r \mathcal{G}$ such that $a = N^r a'$. Because $W_{-r} \mathcal{G} \subset N^r \mathcal{G}$, let $N^r a = b$ for some a. Then by strictness, there exists $a' \in F_\ell \mathcal{G}$ such that $N^r a' = N^r a \in W_{-r} \mathcal{G}$. It follows that $a' \in W_r \mathcal{G}$. Indeed, if $a' \in W_{r+k} \mathcal{G}$ for some $k > 0$ such that $a' \neq 0 \in \mathrm{gr}_{r+k}^W \mathcal{G}$. Then $N^{r+k} a' = 0 \in \mathrm{gr}_{-r-k}^W \mathcal{G}$ because $N^r a' = 0 \in \mathrm{gr}_{-r+k}^W \mathcal{G}$, from which we conclude that $a' \in F_\ell W_{r+k-1} \mathcal{G}$. Thus, iterating the procedure, a' is actually in $F_\ell W_r \mathcal{G}$. We conclude the proof. □

Let $\mathcal{P}_r =_{\mathrm{def}} \ker \left(N^{r+1} : \mathrm{gr}_r^W \mathcal{G} \to \mathrm{gr}_{-r-2}^W \mathcal{G} \right)$ be the primitive part of $\mathrm{gr}^W \mathcal{G}$, which can be identified with

$$\frac{\ker N^{r+1}}{\ker N^r + N \ker N^{r+2}}.$$

See Example 2.7. Recall the Lefschetz decomposition:

$$\mathrm{gr}_r^W \mathcal{G} = \bigoplus_{\ell \ge 0, -\frac{r}{2}} N^\ell \mathcal{P}_{r+2\ell} \text{ for any } r \in \mathbb{Z}.$$

There are two possible ways to define the filtration on $\mathcal{P}_r$: first we have the natural filtration $F_\ell\mathcal{P}_r$ induced from the inclusion $\mathcal{P}_r \to \mathrm{gr}_r^W\mathcal{G}$ and second we can also define the filtration using

$$\frac{F_\ell \ker N^{r+1} + \ker N^r + N\ker N^{r+2}}{\ker N^r + N\ker N^{r+2}}.$$

But indeed, the two different methods result in the same filtration because of the strictness. Let $m \in F_\ell W_r + W_{r-1}$ such that $N^{r+1}m \in W_{-r-3}$ so that represents a class in $F_\ell\mathcal{P}_r$. It suffices to find an element in $F_\ell \ker N^{r+1}$ representing the same class as m in $F_\ell\mathcal{P}_r$. Let $m = m_1 + m_2$ for $m_1 \in F_\ell W_r$ and $m_2 \in W_{r-1}$. It follows that $N^{r+1}m_1 \in F_{\ell+r+1}W_{-r-3}$ because both $N^{r+1}m, N^{r+1}m_2 \in W_{-r-3}$ and $m_1 \in F_\ell W_r$. Since $N^{r+3} : F_{\ell-2}W_{r+3} \to F_{\ell+r+1}W_{-r-3}$ is surjective, there exists $x \in F_{\ell-2}W_{r+3}$ such that $N^{r+3}x = N^{r+1}m_1 \in F_{\ell+r+1}W_{-r-3}$. See the proof of the above theorem. It follows that $m_1 - N^2x \in F_\ell \ker N^{r+1}$ represents the same element as m in $F_\ell\mathcal{P}_r \subset F_\ell\mathrm{gr}_r^W$.

Corollary 5.5. *The Lefschetz decomposition of $\mathrm{gr}^W\mathcal{G}$ respects filtrations, i.e.*

$$F_\bullet\mathrm{gr}_r^W\mathcal{G} = \bigoplus_{\ell\geq 0,-\frac{r}{2}} N^\ell F_{\bullet-\ell}\mathcal{P}_{r+2\ell} \ \textit{for any}\ r \in \mathbb{Z}.$$

Returning to our situation, it follows that:

Theorem 5.6. *The induced operator $R^r : F_\ell\mathrm{gr}_r^W\mathcal{M} \to F_{\ell+r}\mathrm{gr}_{-r}^W\mathcal{M}$ is an isomorphism. Therefore, the Lefschetz decomposition of $\mathrm{gr}^W\mathcal{M}$ respects filtrations, i.e.*

$$F_\bullet\mathrm{gr}_r^W\mathcal{M} = \bigoplus_{\ell\geq 0,-\frac{r}{2}} R^\ell F_{\bullet-\ell}\mathcal{P}_{r+2\ell} \ \textit{for any}\ r \in \mathbb{Z}.$$

5.3. **Identifying the primitive part $\mathcal{P}_r$.** Recall that $Y^J = \cap_{j\in J}Y_j$ for a subset J of the index set I and $\tilde{Y}^{(r+1)}$ is the disjoint union of Y^J such that the cardinality of J is $r+1$. The morphism $\tau^{(r+1)} : \tilde{Y}^{(r+1)} \to X$ is the natural morphism induced by the closed embeddings $\tau^J : Y^J \to X$.

Theorem 5.7. *There exists a canonical filtered isomorphism $\phi_r : (\mathcal{P}_r, F_\bullet\mathcal{P}_r) \to \tau_+^{(r+1)}\omega_{\tilde{Y}^{(r+1)}}(-r)$.*

Proof. Denote by D^J the normal crossing divisor $Y^J \cap Y_{I\smallsetminus J}$ on Y^J. The residue morphism

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} : \Omega_X^{\bullet+n+1}(\log Y)|_Y \to \bigoplus_{\#J=r+1} \Omega_{Y^J}^{\bullet+n-r}(\log D^J)$$

extends to a morphism of complexes of filtered induced $\mathscr{D}_X$-modules

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} : \Omega_X^{\bullet+n+1}(\log Y)|_Y \otimes \mathscr{D}_X \to \bigoplus_{\#J=r+1} \Omega_{Y^J}^{\bullet+n-r}(\log D^J) \otimes \mathscr{D}_X.$$

Denote by $\mathcal{H}^k$ the k-th cohomology $\mathscr{H}^k\left(\Omega_X^{\bullet+n+1}(\log Y)|_Y \otimes \mathscr{D}_X\right)$. Taking 0-th cohomology of the above yields, by Example 2.4

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} : \mathcal{H}^0 \to \bigoplus_{\#J=r+1} \tau_+^J\omega_{Y^J}(*D^J)(-r).$$

Since the morphism $\frac{dt}{t}\wedge : \Omega_{X/\Delta}^{\bullet+n}(\log Y) \to \Omega_X^{\bullet+n+1}(\log Y)$ also extends to the complexes of induced $\mathscr{D}_X$-modules, we have a short exact sequence of $\mathscr{D}_X$-modules

$$0 \longrightarrow \Omega_{X/\Delta}^{\bullet+n}(\log Y)|_Y \otimes \mathscr{D}_X \xrightarrow{\frac{dt}{t}\wedge} \Omega_X^{\bullet+n+1}(\log Y)|_Y \otimes \mathscr{D}_X \longrightarrow \Omega_{X/\Delta}^{\bullet+n+1}(\log Y)|_Y \otimes \mathscr{D}_X \longrightarrow 0.$$

Considering the associated long exact sequence

$$
(5.22)\qquad 0 \longrightarrow \mathcal{H}^{-1} \longrightarrow \mathcal{M} \xrightarrow{R} \mathcal{M} \xrightarrow{\frac{dt}{t}\wedge} \mathcal{H}^0 \longrightarrow 0
$$

we have the morphism $\frac{dt}{t}\wedge : \mathcal{M} \to \mathcal{H}^0$ and it vanishes on the image of R. To motivate the proof, let me do some local calculation. Let $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ represent a local frame of $\Omega^n_{X/\Delta}(\log Y)|_Y$. Then a local section of $\mathcal{M}$ is represented by $\zeta \otimes P$ for P a local section $\mathscr{D}_X$. Then $\mathrm{Res}_{\tilde{Y}^{(r+1)}} \frac{dt}{t} \wedge \zeta \otimes P$ is a section of $\bigoplus_{\#J=r+1} \Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{D}_X$. Post-composing with the projection

$$
\bigoplus_{\#J=r+1} \Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{D}_X \to \bigoplus_{\#J=r+1} \tau^J_+ \omega_{Y^J}(*D^J)(-r),
$$

we make the morphism explicit:

$$
\mathrm{Res}_{\tilde{Y}^{(r+1)}} \circ \frac{dt}{t}\wedge : \mathcal{M} \to \bigoplus_{\#J=r+1} \tau^J_+ \omega_{Y^J}(*D^J)(-r), \quad [\zeta \otimes P] \mapsto [\mathrm{Res}_{\tilde{Y}^{(r+1)}} \frac{dt}{t} \wedge \zeta \otimes P].
$$

Let $\zeta \otimes z_{\overline{J}} P$ represent a class in $\ker R^{r+1}$ for some fixed ordered index subset J with $\#J = r+1$, where $z_{\overline{J}} = \prod_{j \in I \setminus J} z_j$ (Corollary 5.3). Its image under the above morphism only contained in the component $\tau^J_+ \omega_{Y^J}(*D^J)(-r)$ because $z_{\overline{J}}$ vanishes on other components. Thus, the image is the class represented by

$$
(5.23)\qquad \mathrm{Res}_{\tilde{Y}^{r+1}} \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n \otimes z_{\overline{J}} P = \pm \frac{dz_{\overline{J}}}{z_{\overline{J}}} \wedge dz_{k+1} \wedge \cdots dz_n \otimes z_{\overline{J}} P \in \Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{D}_X,
$$

where $\frac{dz_{\overline{J}}}{z_{\overline{J}}} = \bigwedge_{j \in I \setminus J} \frac{dz_j}{z_j}$ and the sign depends on the order of J. In fact, from the calculation we see that the image does not have any pole along D^J, so it is contained in the subsheaf consisting of classes represented by $\Omega^{n-r}_{Y^J} \otimes \mathscr{D}_X$. This means that the class of (5.23) in $\tau^J_+ \omega_{Y^J}(*D^J)(-r)$ is also contained in the image of the inclusion

$$
\tau^J_+ \omega_{Y^J}(-r) \to \tau^J_+ \omega_{Y^J}(*D^J)(-r), \quad [dz_{\overline{J}} \wedge dz_{k+1} \wedge \cdots dz_n \otimes P] \mapsto [\frac{dz_{\overline{J}}}{z_{\overline{J}}} \wedge dz_{k+1} \wedge \cdots dz_n \otimes z_{\overline{J}} P].
$$

See Example 2.4. It follows that we obtain a factorization $\rho_r : \ker R^{r+1} \to \tau^{(r+1)}_+ \omega_{Y^{(r+1)}}(-r)$. In conclusion, we have the following commutative diagram.

$$
\begin{array}{ccc}
\ker R^{r+1} & \xrightarrow{\rho_r} & \tau^{(r+1)}_+ \omega_{Y^{(r+1)}}(-r) \\
\downarrow & & \downarrow \\
\mathcal{M} \xrightarrow{\frac{dt}{t}\wedge} \mathcal{H}^0 & \xrightarrow{\mathrm{Res}} & \bigoplus_{\#J=r+1} \tau^J_+ \omega_{Y^J}(*D^J)(-r)
\end{array}
$$

For a local section $\zeta \otimes z_K P$ where $z_K = \prod_{i \in K} z_i$ a monomial of degree $k - r + 1$, representing a class in $\ker R^r$, its image under ρ_r is indeed zero because z_K annihilates all $\Omega^{n-r}_{Y^J}(\log D^J)$ for index subset J such that $\#J = r+1$. This implies the morphism ρ_r kills $\ker R^r$. The morphism ρ_r also kills $R \ker R^{r+2}$, because by (5.22) $\frac{dt}{t}\wedge$ vanishes on the image of R. Thus it factors through

$$
\phi_r : \mathcal{P}_r = \frac{\ker R^{r+1}}{\ker R^r + R \ker R^{r+2}} \to \tau^{(r+1)}_+ \omega_{Y^{(r+1)}}(-r).
$$

The morphism ϕ_r is filtered surjective because for $dz_{\bar{J}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes P \in \Omega^{n-r}_{Y^J} \otimes F_\ell \mathscr{D}_X$ representing a class in $F_\ell \tau_+^J \omega_{Y^J}(-r)$ with $\#J = r+1$, we can find a lifting class represented by $\zeta \otimes z_{\overline{J}} P$ in $F_\ell \ker R^{r+1}$. It follows that

$$cc(\mathcal{P}_r) \geq cc(\tau_+^{(r+1)} \omega_{Y^{(r+1)}}) = \sum_{\#J=r+1} [T^*_{Y^J} X] \,.$$

Summing up the inequalities gives

$$\sum_{r\geq 0} (r+1) cc(\mathcal{P}_r) \geq \sum_{r \geq 0} (r+1) \sum_{\#J=r+1} [T^*_{Y^J} X] = \sum_{J \subset I} (\#J) \, [T^*_{Y^J} X] \,.$$

On the other hand, by the Lefschetz decomposition and Theorem 4.7, we have

$$\sum_{J\subset I} (\#J) \, [T^*_{Y^J} X] = cc(\mathcal{M}) = cc(\mathrm{gr}^W \mathcal{M}) = \sum_{r\geq 0} (r+1) cc(\mathcal{P}_r).$$

Therefore, all inequalities must be equalities, i.e. $cc(\mathcal{P}_r) = cc(\tau_+^{(r+1)} \omega_{\tilde{Y}^{(r+1)}})$. It follows that ϕ_r is a filtered isomorphism [HTT08, Proposition 3.1.2]. □

6. Reduced case: Sesquilinear pairing on $\mathcal{M}$ and limiting mixed Hodge structure

6.1. **Sesquilinear pairing.** We begin to construct the last data we need for the limiting mixed Hodge structure – Sesquilinear pairing. In the sense that $\mathcal{M}$ is the specialization of $i_{X_t+}\omega_{X_t}$ for $t \neq 0$, the sesquilinear $S : \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ should also be the specialization of $i_{X_t+} S_{X_t}$, where S_{X_t} is defined in §2. Presumably one would expect that the pairing

$$\begin{aligned}\langle S([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta \rangle &= \lim_{t\to 0} \langle i_{X_t+} S_{X_t}(\zeta_1 \otimes P_1, \zeta_2 \otimes P_2), \eta \rangle \\ &= \lim_{t \to 0} \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1 \overline{P_2} \eta \wedge \zeta_1 \wedge \overline{\zeta_2}\end{aligned}$$

should work on $\mathcal{M}$ for $\zeta_i \otimes P_i$, $i = 1, 2$ sections of $\Omega^n_{X/\Delta} \otimes \mathscr{D}_X$ over local chart U representing classes of $\mathcal{M}$, and η is a test function over U. But one could check that the integral $\int_{X_t} P_1 \overline{P_2}\,(\eta)\, \zeta_1 \overline{\wedge \zeta_2}$ could have order $(-\log |t|^2)^k$ near the origin where $k+1$ is the number of components that intersect in U, so the limit may not exist. To avoid the issue, we use a Mellin transform device (see [Sab02, 4.E]): locally

$$\begin{aligned}\langle S([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta \rangle &=_{\mathrm{def}} \mathrm{Res}_{s=0} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} P_1 \overline{P_2} \eta \frac{dt}{t} \wedge \zeta_1 \wedge \overline{\frac{dt}{t} \wedge \zeta_2} \\ &= \mathrm{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \overline{\frac{dt}{t}} \left(\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1 \overline{P_2} \eta \wedge \zeta_1 \wedge \overline{\zeta_2} \right) \\ &= \mathrm{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \overline{\frac{dt}{t}} \langle i_{X_t+} S_{X_t}(\zeta_1 \otimes P_1, \zeta_2 \otimes P_2), \eta \rangle.\end{aligned}$$

The last expression in the definition in some extent explains that S is the specialization of $i_{X_t+} S_{X_t}$ and the 0-current $\mathrm{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \overline{\frac{dt}{t}}$ is doing the job of renormalization of $i_{X_t+} S_{X_t}$ for $t \neq 0$. In fact, for any test function g on Δ, we have

$$\mathrm{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \overline{\frac{dt}{t}} g = g(0).$$

We have not check that S is well-defined, but let us do an example to see how the Mellin transform works.

Example 6.1. Suppose Y is smooth, then R is identical zero and $\mathcal{M} \simeq i_{Y+}\omega_Y$, by Theorem 5.7. Thus, the pairing S should recover the natural pairing S_Y. In local coordinates $t = z_0$ and for any local sections $\zeta_i \otimes P_i = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes P_i$ of $\Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$, $i = 1, 2$ over local chart U,

$$\begin{aligned}\langle S([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle &= \operatorname{Res}_{s=0} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} P_1\overline{P_2}(\eta) \frac{dt}{t} \wedge \zeta_1 \wedge \overline{\frac{dt}{t} \wedge \zeta_2} \\ &= \operatorname{Res}_{s=0} \int_X |t|^{2s-2} P_1\overline{P_2}(\eta) \bigwedge_{i=0}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i} \\ \text{integration by parts on } t \text{ and } \bar{t} &= \operatorname{Res}_{s=0} \int_X \frac{|t|^{2s}}{s^2} \partial_0\overline{\partial_0}\left(P_1\overline{P_2}(\eta)\right) \bigwedge_{i=0}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i}.\end{aligned}$$

Because the Laurent expansion of $s^{-2}|t|^{2s}$ is $\sum_{\ell=0}^{\infty} \left(\log|t|^2\right)^{\ell} s^{\ell-2}$, the above continuously equals to, by Poincaré-Lelong equation [GH14, Page 388]

$$\begin{aligned}\int_X \log|t|^2 \partial_0\overline{\partial_0}\left(P_1\overline{P_2}(\eta)\right) \bigwedge_{i=0}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i} &= \int_Y P_1\overline{P_2}(\eta) \bigwedge_{i=1}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i} \\ &= \frac{\varepsilon(n+1)}{(2\pi\sqrt{1})^n} \int_Y P_1\overline{P_2}(\eta)\zeta_1 \wedge \overline{\zeta_2} \\ &= \langle i_{Y+}S_Y([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle.\end{aligned}$$

We can take a cleaner point of view. In the case Y is smooth, the form $P_1\overline{P_2}(\eta)\zeta_1\overline{\wedge\zeta_2}$ is smooth in the neighborhood of Y. It follows that $i_{X_t+}S_{X_t}$ extends smoothly to $t = 0$ and the limit of $i_{X_t+}S_{X_t}$ is exactly $i_{Y+}S_Y$.

When Y has several smooth irreducible components, the idea of computation is similar to the above. Now we begin to establish the statements needed to ensure S is well-defined. For any test function η over an arbitrary open subset U of X and two sections m_1, m_2 in $H^0\left(U, \Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$, the $(2n+2)$-form $\frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\eta)$ is smooth away from Y but with poles along Y supported in U. Locally, say $m_i = \zeta \otimes P_i$ for $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n$ and $i = 1, 2$, the $(2n+2)$-form $\frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\eta)$ is just $P_1\overline{P_2}(\eta)\frac{dt}{t} \wedge \zeta \wedge \overline{\frac{dt}{t} \wedge \zeta}$. Let $F(s) = F(s, m_1, m_2, \eta)$ be the meromorphic continuation via integration by parts of the following function

$$\frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} \frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\eta).$$

The function $F(s)$ is holomorphic when $\operatorname{Re} s > 0$ and has potential poles at non-positive integers. Note that $F(s)$ is independent of local coordinates. We are only interested in the polar part of the function $F(s)$ at $s = 0$.

Theorem 6.2. *The polar part of $F(s)$ at $s = 0$ only depends on the classes of m_1 and m_2 in $\mathcal{M}$.*

Proof. Let $\{\rho_\lambda\}$ be a partition of unity of the open covering $\{U_\lambda\}$ by local charts. Then

$$F(s) = \sum_\lambda \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_{U_\lambda} |t|^{2s} \frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\rho_\lambda\eta).$$

Since $\rho_\lambda\eta$ is a test function over U_λ, without loss of generality, we can assume U itself is a local chart. It follows that we can assume that $m_i = \zeta \otimes P_i$ for $i = 1, 2$ and $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n$. We begin with some properties of $F(s)$.

Lemma 6.3. *Under the assumption that $m_i = \zeta \otimes P_i$ for $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ and for $i = 1, 2$, the followings are valid.*

(1) the order of the pole of $F(s)$ at $s=0$ is at most $k+1$;
(2) if $P_i = tP_i'$ for one of $i=1,2$, then $F(s)$ is holomorphic at $s=0$;
(3) for $0 \le j \le k$ we have,

$$F(s,\zeta_1 \otimes P_1, \zeta_2 \otimes z_j\partial_j P_2, \eta) = F(s,\zeta_1 \otimes z_j\partial_j P_1, \zeta_2 \otimes P_2,\eta) = -sF(s,\zeta_1\otimes P_1,\zeta_2\otimes P_2,\eta).$$

Proof of the lemma. The Laurent expansion of $F(s)$ at $s=0$ is

$$\begin{aligned} F(s) &= \int_X |z_I|^{2s-2} P_1\overline{P_2}(\eta) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \quad \text{where } z_I = \prod_{i\in I} z_i \\ &= \int_X \frac{|z_I|^{2s}}{s^{2k+2}} \partial_I \overline{\partial_I} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \quad \text{where } \partial_I = \prod_{i=0}^k \partial_i \\ &= \sum_{\ell=0}^\infty \frac{s^{\ell-(2k+2)}}{\ell!} \int_X \left(\log|z_I|^2\right)^\ell \partial_I\overline{\partial_I} P_1\overline{P_2}(\eta) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}). \end{aligned}$$

The order of the pole at $s=0$ is at most $k+1$: if $\ell < k+1$, the form

$$\left(\log|z_I|^2\right)^\ell \partial_I\overline{\partial_I} P_1\overline{P_2}(\eta) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

is actually exact because one of a_i's must be 0 in the expansion of $\left(\log|z_I|^2\right)^\ell$ into a linear combination of $\prod_{i=0}^k \left(\log|z_i|^2\right)^{a_i}$ with $\sum_{i=0}^k a_i = \ell < k+1$. This proves (1).

Suppose that $P_1 = tP_1'$. Then the function

$$F(s) = \int_X |z_I|^{2s-2} tP_1'\overline{P_2}(\eta) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}).$$

is well-defined at $s=0$ because the form

$$\frac{1}{\overline{z_I}} P_1'\overline{P_2}(\eta) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i})$$

is integrable. The same argument works for the case when $P_2 = tP_2'$. This proves (2).

Now we turn to the last statement

$$\begin{aligned} &F(s,\zeta\otimes P_1, \zeta\otimes \overline{z_j\partial_j} P_2, \eta) \\ =&\frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} \overline{z_j\partial_j}(P_1\overline{P_2}\eta) \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge dz_n \wedge \overline{\frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge dz_n} \\ =& \int_X |z_{I\smallsetminus\{j\}}|^{2s-2} z_j^{s-1} \overline{z_j^s \partial_j} P_1 \overline{P_2}\eta \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i}) \\ \text{integration by part on } dz_j =& -s\int_X |z_I|^{2s-2} P_1\overline{P_2}\eta \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\ =& -sF(s,\zeta\otimes P_1,\zeta\otimes P_2,\eta). \end{aligned}$$

The same argument works for $F(s,\zeta\otimes z_j\partial_j P_1, \zeta\otimes P_2,\eta) = -sF(s,\zeta\otimes P_1,\zeta\otimes P_2,\eta)$. This proves (3). ☐

Returning to the proof of the theorem, if one of $\zeta\otimes P_i$ is $\frac{dz_1}{z_1}\wedge\frac{dz_2}{dz_2}\wedge\cdots\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n \otimes tP_i'$, the above lemma (2) says $F(s)$ is holomorphic. If one of $\zeta\otimes P_i$ is $\frac{dz_1}{z_1}\wedge\frac{dz_2}{dz_2}\wedge\cdots\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n\otimes D_iP$, then the (3) above lemma says $F(s)$ is in fact 0. ☐

For any sections $\alpha,\beta\in\mathcal{M}$, let $\{\rho_\lambda\}$ be a partition of unity of the open covering $\{U_\lambda\}$ by local charts such that α,β lifts to $\tilde{\alpha}_\lambda,\tilde{\beta}_\lambda$ over U_λ in $\Omega^n_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$. The above theorem just says that the pairing $S:\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\to\mathfrak{C}_X$ given by

$$\langle S(\alpha,\beta),\eta\rangle =_{\mathrm{def}} \mathrm{Res}_{s=0}\sum_{\lambda}F(s,\tilde{\alpha}_\lambda,\tilde{\beta}_\lambda,\rho_\lambda\eta)$$

is well-defined and does not depend on the choice of partition of unity. By the above lemma we also have the following.

Corollary 6.4. *The operator R is self-adjoint with respect to S, i.e. $S\circ(R\otimes_{\mathbb{C}}\mathrm{id})=S\circ(\mathrm{id}\otimes_{\mathbb{C}}R)$.*

Because the self-adjointness, we have induced pairings on the graded quotient $S_r:\mathrm{gr}^W_r\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathrm{gr}^W_{-r}\mathcal{M}}\to\mathfrak{C}_X$ for every integer r. Denote by P_RS_r the pairing

$$S_r\circ(\mathrm{id}\otimes_{\mathbb{C}}R^r):\mathcal{P}_r\otimes_{\mathbb{C}}\overline{\mathcal{P}_r}\to\mathfrak{C}_X.$$

Theorem 6.5. *The isomorphism $\phi_r:(\mathcal{P}_r,F_\bullet\mathcal{P}_r)\to\tau_+^{(r+1)}\omega_{\tilde{Y}^{(r+1)}}(-r)$ in Theorem 5.7 respects the sesquilinear pairings up to a constant $(-1)^r(r+1)!^{-1}$, i.e.*

$$P_RS_r(\alpha,\beta)=\frac{(-1)^r}{(r+1)!}\tau_+^{(r+1)}S_{\tilde{Y}^{(r+1)}}(\phi_r\alpha,\phi_r\beta)$$

for any local sections $\alpha,\beta\in\mathcal{P}_r$.

Proof. Because the problem is local, it suffices to prove the theorem for α and β are represented by

$$\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n\otimes z_{K_i}$$

and $\#K_i=k-r$ for $i=1,2$ over a local chart U respectively. Recall that $z_K=\prod_{j\in K}z_j$. Let η be a test function over U. We have

$$\langle P_RS_r(\alpha,\beta),\eta\rangle=\langle S(\alpha,R^r\beta),\eta\rangle=\mathrm{Res}_{s=0}(-s)^r\int_X|z_I|^{2s-2}z_{K_1}\overline{z_{K_2}}(\eta)\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}).$$

If $\alpha\neq\beta$, the above is in fact zero. Indeed, for $v\in K_2\smallsetminus K_1$, by choosing $R^r=\prod_{i\in I\smallsetminus K_1\smallsetminus\{v\}}z_i\partial_i$,

$$\langle P_RS_r(\alpha,\beta),\eta\rangle=\langle S(R^r\alpha,\beta),\eta\rangle=\mathrm{Res}_{s=0}\int_X|z_I|^{2s-2}\frac{t}{z_v}\overline{z_v}\tilde{\eta}\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}),$$

where $\tilde{\eta}=\partial_{I\smallsetminus(K_1\smallsetminus\{v\})}\overline{z_{K_2}}(\overline{z_v})^{-1}\eta$ is a smooth test function. The function

$$\int_X|z_I|^{2s-2}\frac{t}{z_v}\overline{z_v}\tilde{\eta}\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

is holomorphic at $s=0$ because

$$\frac{1}{\overline{z_I}}\frac{\overline{z_v}}{z_v}\tilde{\eta}\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

is integrable.

Therefore, we reduce the proof to the case when $\alpha=\beta$ represented by

$$\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge\cdots\wedge dz_n\otimes z_K.$$

We shall prove that

$$P_R S_r(\alpha,\alpha) = \frac{(-1)^r}{(r+1)!}\tau_+^{\overline{K}} S_{Y^{\overline{K}}}(\phi_r\alpha,\phi_r\alpha),$$

where $\overline{K}$ is the complement of K in I. Without loss of generality, we can assume $K = \{r+1, r+2, ..., k\}$. Then

$$\begin{aligned}
P_R S_r(\alpha,\alpha) &= \mathrm{Res}_{s=0}(-s)^r \int_X |z_{\overline{K}}|^{2s-2} \prod_{j=r+1}^{k} |z_j|^{2s} \eta \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
&= (-1)^r \mathrm{Res}_{s=0} s^{-(r+2)} \int_X \prod_{i=0}^{k} |z_i|^{2s} \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \text{ where } \partial_{\bar{K}} = \prod_{i=0}^{r} \partial_i \\
&= \frac{(-1)^r}{(r+1)!} \int_X \left(\log \prod_{i=0}^{k} |z_i|^2\right)^{r+1} \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
(\star) &= \frac{(-1)^r}{(r+1)!} \int_X \prod_{i=0}^{r} \log |z_i|^2 \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
&= \frac{(-1)^r}{(r+1)!} \int_{Y^{\overline{K}}} \eta \bigwedge_{i=r+1}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \quad \text{(Poincaré-Lelong equation [GH14, Page 388])} \\
&= \frac{(-1)^r}{(r+1)!} \tau_+^{\overline{K}} S_{Y^{\overline{K}}}(\mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha, \mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha).
\end{aligned}$$

The equality $(\star)$ holds because if we expand $\left(\log \prod_{i=0}^{k} |z_i|^2\right)^{r+1}$ as a linear combination of $\prod_{i=0}^{k} \left(\log |z_i|^2\right)^{a_i}$ with $\sum_{i=0}^{k} a_i = r+1$, the only possible non-exact form among

$$\prod_{i=0}^{k} \left(\log |z_i|^2\right)^{a_i} \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}),$$

is $\left(\prod_{i=0}^{r} \log |z_i|^2\right) \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \wedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$. Note that while $\mathrm{Res}_{Y^{\overline{K}}}$ depends on the order of the index sets K and I, the pairing

$$\frac{(-1)^r}{(r+1)!} \tau_+^{(r+1)} S_{\tilde{Y}^{(r+1)}}(\phi_r\alpha, \phi_r\beta) = \frac{(-1)^r}{(r+1)!} \tau_+^{\overline{K}} S_{Y^{\overline{K}}}(\mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha, \mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha)$$

does not because the sign will cancel out. We complete the proof. □

6.2. **Constructure of the limiting mixed Hodge structure.** We are going to show that the triple $(\mathrm{DR}_X\mathcal{M}, F, W)$ gives a mixed Hodge complex. Unlike the $\mathbb{Q}$-mixed Hodge complex considered by Deligne [Del71], where the rational structure is a required input, we do not have this piece of information in our situation. We will redo the Deligne's argument on mixed Hodge complex by sesquilinear pairings. It also worths to point out that the sesquilinear pairing makes one check the first page weight spectral sequence of $\mathrm{DR}_X\mathcal{M}$ is a polarzed bigraded Hodge-Lefschetz structure easier than the case in [GNA90], where they need to decompose the differential d_1 on the first page into a combinatorial differential and a sum of Gysin morphisms.

We first set up the pairing on each page of the weight spectral sequence abstractly. Let $\mathcal{N}$ be a holonomic $\mathscr{D}_Z$-module equipped with a sesquilinear pairing $S : \mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_Z$ on a complex manifold Z. Assume that N has compact support. Let N be a nilpotent operator on $\mathcal{N}$ such that $S \circ (\mathrm{id} \otimes_{\mathbb{C}} N) = S \circ (N \otimes_{\mathbb{C}} \mathrm{id})$. Let $W_\bullet\mathcal{N}$ be the monodromy filtration associated to N on $\mathcal{N}$. Denote by $E_r^{i,j}$ be the weight spectral sequence convergent to $\mathrm{gr}_{-i}^W H^{i+j}(Z, \mathrm{DR}_Z\mathcal{N})$ with $E_1^{i,j} = H^{i+j}(Z, \mathrm{gr}_{-i}^W \mathrm{DR}_Z\mathcal{N})$. By abuse of notation, denote by S_k the induced pairing

$$H^k(Z, \mathrm{DR}_Z\mathcal{N}) \otimes_{\mathbb{C}} \overline{H^k(Z, \mathrm{DR}_Z\mathcal{N})} \to H^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}}) \to H_c^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z) \simeq \mathbb{C}$$

multiplying a sign factor $\varepsilon(k)$. Let a be a local section of $(\mathrm{DR}_Z\mathcal{N})^{-j-1}$ and b be a local section of $(\mathrm{DR}_Z\mathcal{N})^{i}$. Then

$$D(a\otimes_{\mathbb{C}} b) = da\otimes_{\mathbb{C}} b + (-1)^{-j-1}a\otimes_{\mathbb{C}} db$$

for D a differential on $\mathrm{DR}_{Z,\overline{Z}}\mathcal{N}\otimes_{\mathbb{C}}\overline{\mathcal{N}}$. Applying S, we find that

$$DS(a,b) = S(da,b) + (-1)^{-j-1}S(a,db). \tag{6.24}$$

Since the differential d is compatible with the weight filtration, we have an induced pairing $E_1(S)_k$ on the first page $E_1^{i,j}$ of the weight spectral sequence by the pairing

$$H^k(Z,\mathrm{gr}^W_{-i}\mathrm{DR}_Z\mathcal{N})\otimes_{\mathbb{C}}\overline{H^k(Z,\mathrm{gr}^W_i\mathrm{DR}_Z\mathcal{N})}\to H^0(Z,\mathrm{DR}_{Z,\overline{Z}}\mathrm{gr}^W_{-i}\mathcal{N}\otimes_{\mathbb{C}}\overline{\mathrm{gr}^W_i\mathcal{N}})\to H^0(Z,\mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z)$$

multiplying a sign factor $\varepsilon(k)$. Then by equation (6.24) we obtain

$$0 = \varepsilon(-j)E_1(S)_{-j}(d_1a,b) + \varepsilon(-j-1)(-1)^{-j-1}E_1(S)_{-j-1}(a,d_1b),$$

since $DSa\otimes_C\bar{b}$ is cohmologous to zero. Working out the sign, the above is equivalent to

$$E_1(S)_{-j}(d_1a,b) + E_1(S)_{-j-1}(a,d_1b) = 0,$$

i.e. the differential d_1 is skew-symmetrc with respect to $E_1(S)$. It follows that we have an induced pairing on the second page: $E_2(S)_k : E_2^{i,k-i}\otimes\overline{E_2^{-i,-k+i}}\to\mathbb{C}$ since $E_2 = \ker d_1/\mathrm{Im}\, d_1$. Again, it follows from the equation (6.24), the differential d_2 is skew-symmetric with respect to $E_2(S)$. By an inductive argument, we get the induced pairing $E_r(S): E_r\otimes\overline{E_r}\to\mathbb{C}$ on the r-th page of the weight spectral sequence $E_r\otimes\overline{E_r}\to\mathbb{C}$ such that d_r is skew-symmetric with respect to $E_r(S)$ for every $r\geq 1$.

Next, let $L = [\omega]\wedge$ be a Lefschetz operator for a Kähler class $[\omega]\in H^1(Z,\Omega_Z)\cap H^2(Z,\mathbb{R})$ on Z which can be thought as a morphism $L:\mathbb{C}\to\mathbb{C}[2]$ in $\mathbf{D}^b(Z,\mathbb{C})$ and so is $\mathsf{X} = 2\pi\sqrt{-1}L$. Therefore, we obtain a morphism $\mathsf{X}:\mathrm{DR}_Z\mathcal{N}\to\mathrm{DR}_Z\mathcal{N}[2]$. Let us work out the relation between the sesquilinear pairing S_k and the operator X. By funtorailty, we have the following commutative diagram in $\mathbf{D}^b(Z,\mathbb{C})$.

$$\begin{array}{ccccccc}
\mathrm{DR}_{Z,\overline{Z}}\mathcal{N}\otimes_{\mathbb{C}}\overline{\mathcal{N}} & \xrightarrow{S} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z & \xrightarrow{\simeq} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{Db}_Z & \xrightarrow{\simeq} & \mathcal{A}_Z^\bullet\otimes\mathfrak{Db}_Z[2\dim Z] \\
\downarrow{\scriptstyle \mathsf{X}\otimes_{\mathbb{C}}\mathrm{id}} & & \downarrow{\scriptstyle\mathsf{X}} & & \downarrow{\scriptstyle\mathsf{X}} & & \downarrow{\scriptstyle\mathsf{X}} \\
\mathrm{DR}_{Z,\overline{Z}}\mathcal{N}\otimes_{\mathbb{C}}\overline{\mathcal{N}}[2] & \xrightarrow{S[2]} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z[2] & \xrightarrow{\simeq} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{Db}_Z[2] & \xrightarrow{\simeq} & \mathcal{A}_Z^\bullet\otimes\mathfrak{Db}_Z[2\dim Z+2]
\end{array}$$

Similarly, we have $S[2]\circ(\mathrm{id}\otimes_{\mathbb{C}}\mathsf{X}) = \overline{\mathsf{X}}S$. It follows from $\mathsf{X}+\overline{\mathsf{X}} = 0$ on $\mathcal{A}_Z^\bullet\otimes\mathfrak{Db}[2\dim Z]$ that

$$\varepsilon(k)S_k(\mathsf{X}-,-) + \varepsilon(k-2)S_{k-2}(-,-) = 0,\quad \text{i.e. } S_k(\mathsf{X}-,-) = S_{k-2}(-,\mathsf{X}-). \tag{6.25}$$

Returning to our situation, we begin to construct a polarized bigraded Hodge-Lefschetz structure on

$$\mathrm{gr}^W H^\bullet(X,\mathrm{DR}_X\mathcal{M}).$$

Fix a Kähler class $[\omega]$ on X and let $L = [\omega]\wedge:\mathrm{DR}_X\mathcal{M}\to\mathrm{DR}_X\mathcal{M}[2]$ be the Lefschetz operator and $\mathsf{X}_1 = 2\pi\sqrt{-1}L$ as the discussion above. Relabel the first page of the weight spectral sequence by

$$V_{\ell,k} = H^\ell(X,\mathrm{gr}^W_k\mathrm{DR}_X\mathcal{M}) = {}^W E_1^{-k,\ell+k}.$$

Let $V = \bigoplus_{\ell,k\in\mathbb{Z}}V_{\ell,k}$ with filtration $F_\bullet V$ induced by $F_\bullet\mathcal{M}$. Denote by $E_i(R)$ the induced operator by R on ${}^W E_i$ and let $\mathsf{Y}_2 = E_1(R)$. Denote by $S_{\ell,k}$ for $\ell,k\in\mathbb{Z}$, the induced pairing on $V_{\ell,k}\otimes\overline{V_{-\ell,-k}}$

$$H^\ell(X,\mathrm{gr}^W_k\mathrm{DR}_X\mathcal{M})\otimes\overline{H^{-\ell}(X,\mathrm{gr}^W_{-k}\mathrm{DR}_X\mathcal{M})}\to H^0(X,\mathrm{DR}_{X,\overline{X}}\mathrm{gr}^W_k\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathrm{gr}^W_{-k}\mathcal{M}})\to H^0_c(X,\mathrm{DR}_{X,\overline{X}}\mathfrak{C}_X)\simeq\mathbb{C}.$$

multiplying a sign factor $\varepsilon(\ell)$. Let d_1 be the differential of E_1. In terms of relabeling, we have

$$d_1 : (V_{\ell,k}, F_\bullet V_{\ell,k}) \to (V_{\ell+1,k-1}, F_\bullet V_{\ell+1,k-1}).$$

Theorem 6.6. *The tuple $(V, \mathsf{X}_1, \mathsf{Y}_2, F_\bullet V, \bigoplus S_{j,k}, d_1)$ gives a differential polarized bigraded Hodge-Lefschetz structure of central weight n.*

Proof. Let us first check the conditions in Theorem 2.10 one by one. It is clear that two operators $\mathsf{X}_1, \mathsf{Y}_2$ are commute. Moreover, we have $\mathsf{Y}_2 : (V_{\ell,k}, F_\bullet V_{\ell,k}) \to (V_{\ell,k-2}, F_{\bullet+1} V_{\ell,k-2})$ such that

$$\mathsf{Y}_2^k : F_\bullet V_{\ell,k} \to F_{\bullet+k} V_{\ell,-k},$$

is an isomorphism by Theorem 5.6. Denote by $P_{\mathsf{Y}_2} V_{-j,r}$ the Y_2-primitive part $\ker \mathsf{Y}_2^{r+1} \cap V_{-j,r} = H^{-j}(X, \mathrm{DR}_X \mathcal{P}_r)$. It follows from Theorem 5.7 that $(P_{\mathsf{Y}_2} V_{-j,r}, F_\bullet P_{\mathsf{Y}_2} V_{-j,r})$ is filtered isomorphic to $H^{-j}(\tilde{Y}^{(r+1)}, \mathrm{DR}_{\tilde{Y}^{(r+1)}} \omega_{\tilde{Y}^{(r+1)}})(-r)$ via ϕ_r. Therefore, $\mathsf{X}_1 F_\bullet P_{\mathsf{Y}_2} V_{-j,r} \subset F_{\bullet-1} P_{\mathsf{Y}_2} V_{-j+2,r}$ and by Hard Lefschetz,

$$\mathsf{X}_1^j : F_\bullet P_{\mathsf{Y}_2} V_{-j,r} \to F_{\bullet-j} P_{\mathsf{Y}_2} V_{j,r}$$

is an isomorphism. It follows from the Lefschetz decomposition of Y_2 that $\mathsf{X}_1^j : F_\bullet V_{-j,r} \to F_{\bullet-j} V_{j,r}$ is an isomorphism. This proves (pbHL1) in Theorem 2.10. (pbHL2) follows from the equation (6.25).

Because the operator R self-adjoin with respect to S by Corollary 6.4, we have $S_{j,r}(-, \mathsf{Y}_2 -) = S_{j,r+2}(\mathsf{Y}_2 -, -)$. By Theorem 6.5, the morphism ϕ_r identifies $P_{\mathsf{Y}_2} S_{-j,r} =_{\mathrm{def}} S_{-j,r}(-, \mathsf{Y}_2^r -)$ with $\frac{(-1)^r}{(r+1)!} S_{\tilde{Y}^{(r+1)},-j}$. Recall that

$$S_{\tilde{Y}^{(r+1)},j}(a,b) = \frac{\varepsilon(n-r+j+1)}{(2\pi\sqrt{-1})^{n-r}} \int_{\tilde{Y}^{(r+1)}} a \wedge \bar{b}, \text{ for } a \in H^{n-r+j}(\tilde{Y}^{(r+1)}) \text{ and } b \in H^{n-r-j}(\tilde{Y}^{(r+1)}),$$

and that $S_{\tilde{Y}^{(r+1)},j}(\mathsf{X}_1^j -, -)$ is a polarization on $H^{n-r-j}_{\mathrm{prim}}(\tilde{Y}^{(r+1)}, \mathbb{C})$. The bi-primitive part $P_{-j,r} = \ker \mathsf{X}_1^j \cap \ker \mathsf{Y}_2^r \cap V_{-j,r}$ together with the induced filtration $F_\bullet P_{-j,r}$ and the sesquilinear pairing $S_{j,r}(\mathsf{X}_1^j -, (-\mathsf{Y})_2^r -)$ is identified with the polarized Hodge structure $H^{n-r-j}_{\mathrm{prim}}(\tilde{Y}^{(r+1)}, \mathbb{C})(-r)$ via ϕ_r. This proves (pbHL3).

It remains to prove that d_1 is a differential of the bigraded Hodge-Lefschetz structure V. Clearly, we have

$$[d_1, \mathsf{X}_1] = [d_1, \mathsf{Y}_2] = 0$$

because d_1 is induced by the differential of $\mathrm{DR}_X \mathcal{M}$ and d_1 preserves $F_\bullet$. The differential d_1 is skew-symmetric with respect to $\bigoplus_{j,r} S_{j,r}$ is formally follows the discussion at the beginning of this subsection. Thus, we finished checking that d_1 is a differential. ☐

Corollary 6.7. *We have the following*

1. *the Hodge spectral sequence degenerates at ${}_F E_1$,*
2. *the weight spectral sequence degenerates at ${}^W E_2$,*
3. *The tuple $\left(\bigoplus_{\ell \in \mathbb{Z}} \mathrm{gr}^W H^\ell(X, \mathrm{DR}_X \mathcal{M}), F, \mathsf{X}_1, \mathsf{Y}_2\right)$ together with the pairing induced by $\bigoplus S_{j,k}$ is a polarized bigradged Hodge-Lefschetz structure of central weight n.*

Proof. We slightly modify the idea of cohomological mixed Hodge complex in [Del71] for statement (1) and (2). I claim that the k-th weight spectral sequence $V^k_{\ell,r} =_{\mathrm{def}} {}^W E_k^{-r,\ell+r}$ together with the induced filtration $F_\bullet$ and the induced pairing $S^k_{\ell,r} \circ (\mathrm{id} \otimes \mathsf{w}) : V^k_{\ell,r} \otimes \overline{V^k_{\ell,r}} \to \mathbb{C}$ is a polarized Hodge structure of weight $n+\ell+r$ and the differential $d_k : V^k_{\ell,r} \to V^k_{\ell+1,r-k}$ is a morphism of Hodge structures. Indeed, the differential d_k is skew-symmetric with respect to the sesquilinear pairing, i.e. $S^k_{\ell,r}(d_k -, -) + S^k_{\ell+1,r-k}(-, d_k -) = 0$. Therefore, if $(-1)^q S^k_{\ell,r} \circ (\mathrm{id} \otimes \mathsf{w})$ for $q = n+\ell+r-p$ is a Hermitian inner product on

$$\left(V^k_{\ell,r}\right)^{p,q} = \{a \in F^p V^k_{\ell,r} : S^k_{\ell,r}(a,b) = 0 \text{ for all } b \in F^{p-\ell-r+1} V^k_{-\ell,-r}\}$$

then $(-1)^q S_{\ell,r}^{k+1} \circ (\mathrm{id} \otimes \mathsf{w})$ is also a Hermitian inner product on

$$\left(V_{\ell,r}^{k+1}\right)^{p,q} = \{a \in F^p V_{\ell,r}^{k+1} : S_{\ell,r}^{k+1}(a,b) = 0 \text{ for all } b \in F^{p-\ell-r+1} V_{-\ell,-r}^{k+1}\}.$$

In particular, we have the decomposition

$$V_{\ell,r}^{k+1} = \bigoplus_{p+q=n+\ell+r} \left(V_{\ell,r}^{k+1}\right)^{p,q}$$

and the morphism $d_k : \left(V_{\ell,r}^{k}\right)^{p,q} \to \left(V_{\ell,r}^{k+1}\right)^{p,q}$ is compatible with the decomposition. See Remark 2.11. By induction the claim is proved. It follows that d_k vanishes for $k \geq 2$ by it is a morphism of Hodge structures of different weights, which proves (2).

Since each bigraded piece $V_{\ell,r} = H^\ell\left(X, \mathrm{gr}_r^W \mathrm{DR}_X \mathcal{M}\right)$ is pure Hodge structure of weight $n+r+\ell$, the two vector spaces $H^\ell\left(X, \mathrm{gr}^F \mathrm{gr}_r^W \mathrm{DR}_X \mathcal{M}\right)$ and $V_{\ell,r}$ is isomorphic. Moreover, the isomorphism is compatible with d_1, because d_1 respects $F_\bullet$ and

$$\mathrm{gr}_r^W \mathrm{gr}^F \mathrm{DR}_X \mathcal{M} = \mathrm{gr}^F \mathrm{gr}_r^W \mathrm{DR}_X \mathcal{M}.$$

Taking cohomology of d_1, we obtain that $\mathrm{gr}_r^W H^\ell\left(X, \mathrm{gr}^F \mathrm{DR}_X \mathcal{M}\right)$ is isomorphism to $\mathrm{gr}_r^W H^\ell\left(X, \mathrm{DR}_X \mathcal{M}\right)$. It follows from the dimension reason that $H^\ell\left(X, \mathrm{gr}^F \mathrm{DR}_X \mathcal{M}\right)$ is isomorphic to $H^\ell\left(X, \mathrm{DR}_X \mathcal{M}\right)$, which is exactly the degeneration of Hodge spectral sequence at ${}_F E_1$.

The statement (3) follows from Theorem 2.12. □

The third statement in the above corollary ensures that the weight filtration on the hypercohomology of $\mathrm{DR}_X \mathcal{M}$ is the monodromy weight filtration of the nilpotent operator R, i.e. $R W_\bullet H^\ell(X, \mathrm{DR}_X \mathcal{M}) \subset W_{\bullet-2} H^\ell(X, \mathrm{DR}_X \mathcal{M})(-1)$ and $R^r : \mathrm{gr}_r^W H^\ell(X, \mathrm{DR}_X \mathcal{M}) \to \mathrm{gr}_{-r}^W H^\ell(X, \mathrm{DR}_X \mathcal{M})(-r)$ is a filtered isomorphism. We proved Theorem A for the case when Y is reduced.

7. Non-reduced case: Generalized eigenspace $\mathcal{M}_\alpha$ and the weight filtration

Now we move to the general situation. Recall that we have introduced the notations: the index set I consisting of indices of irreducible components of Y and e_i is the multiplicity of Y along the component Y_i.

7.1. The generalized eigen-modules $\mathcal{M}_\alpha$.

We begin with studying the generalized eigen-modules $\ker(R-\alpha)^\infty$ of the morphism R in the category of filtered $\mathscr{D}_X$-modules. The generalized eigen-modules are naturally sub-modules of $\mathcal{M}$ and one can put the induced filtration on it. However, this filtration does not match with the expected weight of the mixed Hodge structure and is difficult to study. Instead, we use the idea of Saito in [Sai90]: one regards the generalized eigen-module as a sub-quotient of $\mathcal{M}$ and puts the induced filtration on it. It turns out the filtration behaves nice. Now let us begin to settle some definitions.

Define $\mathcal{M}_{\geq\alpha} = \ker \prod_{\lambda \geq \alpha} (R-\lambda)^\infty$, $\mathcal{M}_{>\alpha} = \ker \prod_{\lambda>\alpha}(R-\lambda)^\infty$ and $\mathcal{M}_\alpha = \mathcal{M}_{\geq\alpha}/\mathcal{M}_{>\alpha}$. Then $\mathcal{M}_\alpha$ is canonically isomorphic to the generalized eigen-module $\ker(R-\alpha)^\infty$. Endow $\mathcal{M}_\alpha$ the filtration $F_\bullet \mathcal{M}_\alpha$ induced from $(\mathcal{M}, F_\bullet \mathcal{M})$,

$$F_\bullet \mathcal{M}_\alpha = \frac{\mathcal{M}_{\geq\alpha} \cap F_\bullet \mathcal{M}}{\mathcal{M}_{>\alpha} \cap F_\bullet \mathcal{M}}.$$

There are parallel definitions on the relative log de Rham complex. Denote by $C^\bullet = \Omega_{X/\Delta}^{\bullet+n}(\log Y) \otimes \mathscr{O}_Y$ for simplicity. Define sub-complexes of $C^\bullet$ by

$$C_{\geq\alpha}^\bullet = C^\bullet \otimes \mathscr{O}_X(-\lceil \alpha Y \rceil), \quad C_{>\alpha}^\bullet = C^\bullet \otimes \mathscr{O}_X(-\lfloor \alpha Y \rfloor - Y_{\mathrm{Red}}) \quad \text{and } C_\alpha^\bullet = C_{\geq\alpha}^\bullet / C_{>\alpha}^\bullet,$$

where Y_{Red} is the associated reduced divisor of Y. Notice that if we let I_α be the subset of I consisting of all i such that αe_i is an integer, then

$$C_\alpha^\bullet = C_{\geq\alpha}^\bullet \otimes \mathscr{O}_{Y_{I_\alpha}}, \quad \text{where } Y_{I_\alpha} = \sum_{i\in I_\alpha} Y_i.$$

One can check $C_\alpha^\bullet$ is a generalized eigen-perverse sheaves of the residue $[\nabla]$. Since $\mathscr{O}_X(-\lceil\alpha Y\rceil)$ is preserved by relative log differential $\mathscr{T}_{X/\Delta}(-\log Y)$, the multiplication by relative log differentials gives a morphism, recalling that $D_1, D_2, ..., D_n$ are local generators of $\mathscr{T}_{X/\Delta}(-\log Y)$ dual to the local generators $\xi_1, \xi_2, ..., \xi_n$ of $\Omega_{X/\Delta}(\log Y)$,

$$\mathscr{O}_X(-\lceil\alpha Y\rceil)\otimes\mathscr{D}_X \to \Omega_{X/\Delta}(-\lceil\alpha Y\rceil)\otimes\mathscr{D}_X, \quad z_I^{\lceil\alpha\mathbf{e}\rceil}\otimes P \mapsto \sum_j \xi_j\otimes D_j z_I^{\lceil\alpha\mathbf{e}\rceil}\otimes P = \sum_j \xi_j \otimes z_I^{\lceil\alpha\mathbf{e}\rceil}(D_j+\alpha_j)\otimes P, \tag{7.26}$$

where, using the multi-index notation, $z_I^{\lceil\alpha\mathbf{e}\rceil} = \prod_{i\in I} z_i^{\lceil\alpha e_i\rceil}$ denotes the local generator of $\mathscr{O}_X(-\lceil\alpha Y\rceil)$ and define $\alpha_i = [D_i, z_I^{\lceil\alpha\mathbf{e}\rceil}]/z_I^{\lceil\alpha\mathbf{e}\rceil} = \lceil\alpha e_i\rceil/e_i - \lceil\alpha e_0\rceil/e_0$. The morphism extends to a complex $\Omega_{X/\Delta}^{n+\bullet}(\log Y)(-\lceil\alpha Y\rceil)\otimes\mathscr{D}_X$, which is a subcomplex of $\Omega_{X/\Delta}^{n+\bullet}(\log Y)\otimes\mathscr{D}_X$ (see (4.17)). Tensoring $\mathscr{O}_Y$ on the left gives $C_{\geq\alpha}^\bullet\otimes\mathscr{D}_X$ by the above definition. Further tensoring $\mathscr{O}_{Y_{I_\alpha}}$ on the left, we obtain the complex of induced $\mathscr{D}_X$-modules $C_\alpha^\bullet\otimes\mathscr{D}_X$ with the filtration defined by

$$F_\ell\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right) = C_\alpha^\bullet\otimes F_{\ell+n+\bullet}\mathscr{D}_X.$$

The following two theorems give the description of the generalized eigen-modules in terms of complexes of the induced $\mathscr{D}_X$-modules.

Theorem 7.1. *The complex $C_\alpha^\bullet\otimes\mathscr{D}_X$ is filtered acyclic and the characteristic cycle of the 0-th cohomology is*

$$cc\left(\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)\right) = \sum_{J\subset I}\left(\#I_\alpha\cap J\right)\left[T^*_{Y^J}X\right].$$

Proof. Similarly to the proof of Theorem 4.1 and Theorem 4.5, the associated graded $\mathrm{gr}^F\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)$ locally is the Koszul complex of the regular sequence $(t_\alpha, D_1, D_2, ..., D_n)$ over $\mathrm{gr}^F\mathscr{D}_X$, where $t_\alpha = \prod_{i\in I_\alpha} z_i$ is the defining equation of Y_{I_α}. It follows that $\mathrm{gr}^F(C_\alpha^\bullet\otimes\mathscr{D}_X)$ is acyclic and therefore, $C_\alpha^\bullet\otimes\mathscr{D}_X$ is filtered acyclic. We also get that $\mathrm{gr}^F\mathscr{H}^0(C_\alpha^\bullet\otimes\mathscr{D}_X)$ is locally represented by

$$\zeta_\alpha\otimes\mathrm{gr}^F\mathscr{D}/(t_\alpha, D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X, \quad \text{where } \zeta_\alpha = z_I^{\lceil\alpha\mathbf{e}\rceil}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n. \tag{7.27}$$

As the calculation in Theorem 4.7, we get the characteristic cycle is $\sum_{J\subset I}\left(\#I_\alpha\cap J\right)\left[T^*_{Y^J}X\right]$. ☐

Theorem 7.2. *There exists a canonical filtered isomorphism*

$$\psi_\alpha : \left(\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right), F_\bullet\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)\right) \xrightarrow{\sim} (\mathcal{M}_\alpha, F_\bullet\mathcal{M}_\alpha). \tag{7.28}$$

In particular, the characteristic cycle $cc(\mathcal{M}_\alpha) = \sum_{J\subset I}\left(\#I_\alpha\cap J\right)\left[T^*_{Y^J}X\right]$.

We first study $\mathcal{M}_{\geq\alpha}$ and $\mathcal{M}_{>\alpha}$ locally by pointing out their cyclic generator. In principal, this always can be done because every holonomic $\mathscr{D}_X$-module locally is cyclic.

Lemma 7.3. *Locally, $\mathcal{M}_{\geq\alpha}$ is generated by $z_I^{\lceil\alpha\mathbf{e}\rceil}$, and $\mathcal{M}_{>\alpha}$ is generated by $z_I^{\lfloor\alpha\mathbf{e}\rfloor+\mathbf{1}}$ where $\mathbf{1} = (1,1,...,1)\in\mathbb{Z}^I$.*

Proof. Let us first check that $z_I^{\lceil\alpha\mathbf{e}\rceil}\in\mathcal{M}_{\geq\alpha}$. It suffices to check that it is in

$$\ker\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(R-\frac{j}{e_i}).$$

This is follows from direct calculation:

$$\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(R-\frac{j}{e_i})z_I^{\lceil\alpha\mathbf{e}\rceil}=\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(R-\frac{j}{e_i})z_i^{\lceil\alpha e_i\rceil}=\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(\frac{1}{e_i}z_i\partial_i-\frac{j}{e_i})z_i^{\lceil\alpha e_i\rceil}$$
$$=\prod_{i\in I}\frac{1}{e_i^{e_i-\lceil\alpha e_i\rceil}}z_i^{e_i}\partial_i^{e_i-\lceil\alpha e_i\rceil}=t\prod_{i\in I}\frac{1}{e_i^{e_i-\lceil\alpha e_i\rceil}}\partial_i^{e_i-\lceil\alpha e_i\rceil}=0\in\mathcal{M}.$$

Because R satisfies the identity (4.19), $\mathcal{M}_{\geq\alpha}$ is also equal to the image of $\prod_{i\in I}\prod_{j=0}^{\lceil\alpha e_i\rceil-1}(R-\frac{j}{e_i})$. It follows from

$$\prod_{i\in I}\prod_{j=0}^{\lceil\alpha e_i\rceil-1}(R-\frac{j}{e_i})(1)=\prod_{i\in I}\prod_{j=0}^{\lceil\alpha e_i\rceil-1}(\frac{1}{e_i}z_i\partial_i-\frac{j}{e_i})=z_I^{\lceil\alpha\mathbf{e}\rceil}\prod_{i\in I}\frac{1}{e_i^{\lceil\alpha e_i\rceil}}\partial_i^{\lceil\alpha e_i\rceil}$$

that $z_I^{\lceil\alpha\mathbf{e}\rceil}\prod_{i\in I}\partial_i^{\lceil\alpha e_i\rceil}$ generates $\mathcal{M}_{\geq\alpha}$. We deduce that $z_I^{\lceil\alpha\mathbf{e}\rceil}$ generates $\mathcal{M}_{\geq\alpha}$. The similar argument works for $\mathcal{M}_{>\alpha}$. □

Proof of Theorem 7.2. It follows from the above lemma that $\mathcal{M}_\alpha$ is locally isomorphic to

$$\zeta\otimes\left(z_I^{\lceil\alpha\mathbf{e}\rceil},D_1,D_2,...,D_n\right)\mathscr{D}_X/\left(z_I^{\lfloor\alpha\mathbf{e}\rfloor+\mathbf{1}},D_1,D_2,...,D_n\right)\mathscr{D}_X$$

where $\zeta=\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge\cdots\wedge dz_n$ so that $\zeta_\alpha=z_I^{\lceil\alpha\mathbf{e}\rceil}\zeta$. Since $\mathscr{H}^0(C_\alpha^\bullet\otimes\mathscr{D}_X)$ by (7.26) is locally isomorphic to

$$\zeta_\alpha\otimes\mathscr{D}_X/(t_\alpha,D_1+\alpha_1,D_2+\alpha_2,...,D_n+\alpha_n)\mathscr{D}_X,$$

the multiplication $\mathscr{H}^0(C_\alpha^\bullet\otimes\mathscr{D}_X)\to\mathcal{M}_\alpha$, $\zeta_\alpha\otimes P\mapsto\zeta\otimes z_I^{\lceil\alpha\mathbf{e}\rceil}P$ is well-defined, does not depend on the coordinate and therefore, gives a filtered morphism

$$\psi_\alpha:(\mathscr{H}^0(C_\alpha^\bullet\otimes\mathscr{D}_X),F_\bullet\mathscr{H}^0(C_\alpha^\bullet\otimes\mathscr{D}_X))\longrightarrow(\mathcal{M}_\alpha,F_\bullet\mathcal{M}_\alpha).$$

The surjectivity is clear from the local description. It follows that $cc\left(\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)\right)\geq cc(\mathcal{M}_\alpha)$. Summing over all the rational numbers α in $[0,1)$ gives

$$\sum_\alpha cc\left(\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)\right)\geq\sum_\alpha cc(\mathcal{M}_\alpha)=cc(\mathcal{M}).$$

On the other hand, by Theorem 4.5 and Theorem 7.1, the $\mathscr{D}_X$-module $\mathcal{M}$ is also successive extensions of $\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)$ for $\alpha\in\mathbb{Q}\cap[0,1)$. Thus,

$$\sum_\alpha cc\left(\mathscr{H}^0\left(C_\alpha^\bullet\otimes\mathscr{D}_X\right)\right)=cc(\mathcal{M}).$$

This forces that ψ_α must be isomorphism and therefore, filtered injective.

It remains to show that

$$F_\ell\psi_\alpha:F_\ell\mathscr{H}^0(C_\alpha^\bullet\otimes\mathscr{D}_X)\longrightarrow F_\ell\mathcal{M}_\alpha, \tag{7.29}$$

is sujective. Suppose that $z_I^{\lceil\alpha\mathbf{e}\rceil}P\in\mathscr{D}_X$ is a representative of a class in $F_\ell\mathcal{M}_\alpha$. Then we can write

$$z_I^{\lceil\alpha\mathbf{e}\rceil}P=P'+\sum_{i=1}^n D_iQ_i+z_I^{\lfloor\alpha\mathbf{e}\rfloor+\mathbf{1}}T$$

for $P'\in F_{\ell+n}\mathscr{D}_X$ and $T,Q_i\in\mathscr{D}_X$. It follows that

$$z_I^{\lceil\alpha\mathbf{e}\rceil}(P-t_\alpha T)=P'+\sum_{i=1}^n D_iQ_i$$

By the regular sequence argument of Theorem 4.5, we can assume that $P - t_\alpha T$ is in $F_{\ell+n}\mathscr{D}_X$. Then the class represented by $P - t_\alpha T$ in $\mathscr{H}^0(C^\bullet_\alpha \otimes \mathscr{D}_X)$ is actually in $F_\ell\mathscr{H}^0(C^\bullet_\alpha \otimes \mathscr{D}_X)$ by the local formula. Therefore, we find a lifting represented by P in $F_\ell\mathscr{H}^0(C^\bullet_\alpha \otimes \mathscr{D}_X)$ of the class of $z_I^{\lceil \alpha \mathbf{e}\rceil}P$ in $F_\ell\mathcal{M}_\alpha$. We conclude the proof. □

Without loss of generality, we can assume by abuse of notation that locally $I_\alpha = \{0,1,\dots,\mu\}$ so that $t_\alpha = z_0 z_1 \cdots z_\mu$. Let R_α be the induced operator $(R-\alpha)$ on $(\mathcal{M}_\alpha, F_\bullet\mathcal{M}_\alpha)$. One easily gets a nice local formula of R_α:

Corollary 7.4. *The endormorphism R_α of $\mathcal{M}_\alpha$ acts locally as $\psi_\alpha \circ (\mathrm{id} \otimes \frac{1}{e_j} z_j\partial_j) \circ (\psi_\alpha)^{-1}$ for any $j \in I_\alpha$.*

Proof. Because $R-\alpha$ acts on the left hand side of the identification (7.27) by the left multiplication by $\frac{1}{e_0}z_0\partial_0 - \alpha$, the statement follows from

$$\begin{aligned} R_\alpha\left[\zeta \otimes z_I^{\lceil \alpha \mathbf{e}\rceil}\right] &= \left[\zeta \otimes \left(\frac{1}{e_j}z_j\partial_j - \alpha\right)\left(z_I^{\lceil \alpha \mathbf{e}\rceil}\right)\right] \\ &= \left[\zeta \otimes \left(\left(\frac{1}{e_j}\lceil \alpha e_j\rceil - \alpha\right) z_I^{\lceil \alpha \mathbf{e}\rceil} + z_I^{\lceil \alpha \mathbf{e}\rceil}\left(\frac{1}{e_j}z_j\partial_j\right)\right)\right] \\ &= \psi_\alpha\left[\zeta z_I^{\lceil \alpha \mathbf{e}\rceil} \otimes \left(\frac{1}{e_j}z_j\partial_j\right)\right] = \psi_\alpha \circ \left(\mathrm{id} \otimes \frac{1}{e_j}z_j\partial_j\right) \circ \psi_\alpha^{-1}\left[\zeta_\alpha \otimes 1\right]. \end{aligned}$$

This completes the proof. □

By the local formula of R_α, it is obvious that $R_\alpha : (\mathcal{M}_\alpha, F_\bullet\mathcal{M}_\alpha) \to (\mathcal{M}_\alpha, F_{\bullet+1}\mathcal{M}_\alpha)$ is a filtered morphism.

7.2. Striness of R_α.

Similar to the reduced case, every power of R_α is strict.

Theorem 7.5. *The power of the endomorphism R_α on $(\mathcal{M}_\alpha, F_\bullet\mathcal{M}_\alpha)$ is strict:*

$$R_\alpha^a F_b\mathcal{M}_\alpha = F_{a+b}R_\alpha^a\mathcal{M}_\alpha, \quad \textit{for any } a \in \mathbb{Z}_{\geq 0} \textit{ and } b \in \mathbb{Z}. \tag{7.30}$$

Let $[R_\alpha]$ be the endomorphism on $\mathrm{gr}^F\mathcal{M}_\alpha$ induced by R_α. To prove the above theorem, we need the following statement on $\ker[R_\alpha] \subset \mathrm{gr}^F\mathcal{M}_\alpha$.

Lemma 7.6. $\ker[R_\alpha]^{r+1}$ *is locally generated by monomials of degree $\mu - r$ that divid t_α.*

Proof of Theorem 7.5. Temporarily admitting this lemma, let $R_\alpha^{r+1}m$ be an element in $F_{\ell+r+1}\mathcal{M}_\alpha$. Assume that $m \in F_k\mathcal{M}_\alpha$. If $k > \ell$ then the projection of $R_\alpha^{r+1}m$ vanishes in $\mathrm{gr}^F_{k+r+1}\mathcal{M}_\alpha$. It follows from the lemma that m can be written as

$$m = \sum_{\substack{\#J=\mu-r,\\ J\subset I_\alpha}} z_J m_J + \sum_{i=1}^n D_iQ_i + m', \qquad \text{for } z_J = \prod_{j\in J} z_j$$

where $Q_i, m' \in F_{k-1}\mathcal{M}_\alpha$. Because for every $J \subset I_\alpha$ of cardinality $r+1$ we can arrange

$$R_\alpha^{r+1}z_J = \prod_{j\in I_\alpha\setminus J}\frac{1}{e_j}z_j\partial_j z_J = t_\alpha\prod_{j\in I_\alpha}\frac{1}{e_j}\partial_j = 0 \in \mathcal{M}_\alpha$$

it follows that $R_\alpha^{r+1}m$ is equal to,

$$\begin{aligned} \sum_{\substack{\#J=\mu-r,\\ J\subset I_\alpha}} R_\alpha^{r+1}z_J m_J + R_\alpha^{r+1}\left(\sum_{i=1}^n D_iQ_i + m'\right) &= \sum_{\substack{\#J=\mu-r,\\ J\subset I_\alpha}} t_\alpha m'_J + \sum_{i=1}^n (D_i+\alpha)R_\alpha^{r+1}Q_i + R_\alpha^{r+1}\left(m' - \sum_{i=1}^n \alpha Q_i\right) \\ &= R_\alpha^{r+1}\left(m' - \sum_{i=1}^n \alpha Q_i\right) \in \mathcal{M}_\alpha. \end{aligned}$$

But now $m' - \sum_{i=1}^{n} \alpha Q_i \in F_{k-1}\mathcal{M}_\alpha$. Iterating the above argument one can find $\tilde{m} \in F_\ell \mathcal{M}_\alpha$ such that

$$R_\alpha^{r+1} m = R_\alpha^{r+1} \tilde{m}.$$

This completes the proof of the theorem. □

Proof of the lemma. The proof is essentially the same as the reduced case. Note that we are now working over the commutative ring $\mathrm{gr}^F \mathscr{D}_X$. We prove by induction on r. Let $P \in \mathrm{gr}^F \mathscr{D}_X$ represent an element of $\ker[R_\alpha]^{r+1}$. When $r = 0$, we have

$$\frac{1}{e_0} z_0 \partial_0 P = t_\alpha Q_0 + \sum_{i=1}^{n} D_i Q_i \quad \text{recalling that } t_\alpha = z_0 z_1 \cdots z_\mu. \tag{7.31}$$

Then $t_\alpha Q_0$ is in the ideal generated by $\partial_0, \partial_1, ..., \partial_\mu, z_{\mu+1}\partial_{\mu+1}, z_{\mu+2}\partial_{\mu+2}, ..., z_k\partial_k, \partial_{k+1}, ...\partial_n$ over $\mathrm{gr}^F \mathscr{D}_X$. Because t_α together with $\partial_0, \partial_1, ..., \partial_\mu, z_{\mu+1}\partial_{\mu+1}, z_{\mu+2}\partial_{\mu+2}, ..., z_k\partial_k, \partial_{k+1}, ...\partial_n$ form a regular sequence in $\mathrm{gr}^F \mathscr{D}_X$, Q_0 can be written as,

$$Q_0 = \sum_{a=0}^{\mu} \partial_a Q_a + \sum_{b=\mu+1}^{k} z_b \partial_b Q_b + \sum_{c=k+1}^{n} \partial_c Q_c.$$

Substuiting in (7.31)

$$\frac{1}{e_0} z_0 \partial_0 \left(P - \sum_{a=0}^{\mu} e_a \frac{t_\alpha}{z_a} Q_a - \sum_{b=\mu+1}^{k} e_b t_\alpha Q_b \right) \in (D_1, D_2, ..., D_n)\mathrm{gr}^F \mathscr{D}_X.$$

Now because $(z_0\partial_0, D_1, D_2, ..., D_n)$ is a regular sequence in $\mathrm{gr}^F \mathscr{D}_X$, P is a linear combination of t_α / z_a for $a \in \{0, 1, ..., \mu\}$ and $D_1, D_2, ..., D_n$ over $\mathrm{gr}^F \mathscr{D}_X$. This concludes the case when $r = 0$.

Assume the statement is true for the case when the exponent is less than r. Because $[R_\alpha]$ sends the class of P to $\ker[R_\alpha]^r$, by induction hypothesis we have

$$\frac{1}{e_0} z_0 \partial_0 P = \sum_{\substack{\#J = \mu - r + 1, \\ J \subset I_\alpha}} z_J Q_J + \sum_{i=1}^{n} D_i Q_i \qquad \text{recalling that } z_J = \prod_{j \in J} z_j. \tag{7.32}$$

Fixing a subset J, then $z_J Q_J$ is in the submodule generated by z_a for $a \in I_\alpha \smallsetminus J$, ∂_b for $b \in J$, $z_c\partial_c$ for $c \in I \smallsetminus I_\alpha$ and ∂_d for $d \notin I$ over $\mathrm{gr}^F \mathscr{D}_X$. Because the elements $z_a, \partial_b, z_c\partial_c, \partial_d$ for $a \in I_\alpha \smallsetminus J, b \in J, c \in I \smallsetminus I_\alpha, d \notin I$ together with z_J form a regular sequence in $\mathrm{gr}^F \mathscr{D}_X$, we deduce that

$$Q_J = \sum_{a \in I_\alpha \smallsetminus J} z_a Q_a + \sum_{b \in J} \partial_b Q_b + \sum_{c \in I \smallsetminus I_\alpha} z_c \partial_c Q_c + \sum_{d \notin I} \partial_d Q_d.$$

Substituting in (7.32), we deduce that

$$\frac{1}{e_0} z_0 \partial_0 \left(P - \left(\sum_{b \in J} e_b \frac{z_J}{z_b} Q_b + \sum_{c \in I \smallsetminus I_\alpha} e_c z_J Q_c \right) \right) - \sum_{a \in I_\alpha \smallsetminus J} z_J z_a Q_a$$

is in the submodule generated by degree $\mu - r + 1$ monomials dividing t_α except z_J and by $D_1, D_2, ..., D_n$ over $\mathrm{gr}^F \mathscr{D}_X$. This means we can reduce $z_J Q_J$ one by one for each J on the right-hand side of the equation (7.32) and at the last step we find that $\frac{1}{e_0} z_0 \partial_0 (P - P')$ is a linear combination of degree $\mu - r + 2$ monomials dividing t_α and $D_1, D_2, ..., D_n$, where P' is a linear combination of degree $\mu - r$ monomials dividing t_α.

Note that the left multiplication by $\frac{1}{e_0} z_0 \partial_0$ has the same effect as applying $[R_\alpha]$ on $\mathrm{gr}^F \mathcal{M}_\alpha$. Therefore, the class represented by $P - P'$ is in $\ker[R_\alpha]^r$ since degree $\mu - r + 2$ monomials dividing t_α is in $\ker[R_\alpha]^{r-1}$. By induction hypothesis the class represented $P - P'$ is a linear combination of degree $\mu - r + 1$ monomials dividing t_α. Therefore, the class represented by P in $\mathrm{gr}^F \mathcal{M}_\alpha$ is a linear combination of degree $\mu - r$ monomials dividing t_α. This completes the proof. □

Corollary 7.7. *The* $\ker R_\alpha^{r+1}$ *is also generated by degree* $\mu - r$ *monomials dividing* t_α *if one identifies* $\mathcal{M}_\alpha$ *locally with* $\mathscr{D}_X/(t_\alpha, D_1, D_2, ..., D_n)\mathscr{D}_X$.

The proof is the same as the one of Corollary 5.3

7.3. **The weight filtration.** Now the weight filtration of each generalized eigen-modules interacts well with the good filtration because of the strictness. Recall that since R_α is nilpotent on $\mathcal{M}_\alpha$, it induces a $\mathbb{Z}$-indexed filtration $W_\bullet \mathcal{M}_\alpha$. We filtered the sub-module $W_r\mathcal{M}_\alpha$ by the induced filtration $F_\bullet W_r \mathcal{M}_\alpha = F_\bullet \mathcal{M}_\alpha \cap W_r \mathcal{M}_\alpha$. Let

$$\mathcal{P}_{\alpha,r} = \frac{\ker R_\alpha^{r+1}}{\ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}}$$

be the r-th primitive part of $\mathrm{gr}^W \mathcal{M}_\alpha$ with the filtration defined by

$$F_\ell \mathcal{P}_{\alpha,r} = \frac{F_\ell \ker R_\alpha^{r+1} + \ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}}{\ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}}.$$

As the formal proof in Theorem 5.6, we have

Corollary 7.8. *The induced operator* $R_\alpha^r : F_\ell \mathrm{gr}_r^W \mathcal{M}_\alpha \to F_{\ell+r}\mathrm{gr}_{-r}^W \mathcal{M}_\alpha$ *is an isomorphism. Therefore, the Lefschetz decomposition of* $\mathrm{gr}^W \mathcal{M}_\alpha$ *respects filtrations, i.e.*

$$F_\bullet \mathrm{gr}_r^W \mathcal{M}_\alpha = \bigoplus_{\ell \geq 0, -\frac{r}{2}} R_\alpha^\ell F_{\bullet-\ell} \mathcal{P}_{\alpha,r+2\ell} \text{ for any } r \in \mathbb{Z}.$$

7.4. **Summands of the primitive part $\mathcal{P}_{\alpha,r}$.** Recall that $Y^J = \bigcap_{j\in J} Y_j$ and $Y_J = \bigcup_{j \in J} Y_j$ for any subset J of I and e_j is the multiplicity of Y_j in Y. Like the reduced case that $\mathcal{P}_r$ decomposes into the direct images of $\omega_{Y^J}(-r)$ for all index subset sJ of cardinality $r+1$ (Theorem 5.7), the primitive part $\mathcal{P}_{\alpha,r}$ of the generalized α-eigemodule also decomposes into direct images of certain filtered $\mathscr{D}_{Y^J}$-modules $\mathcal{V}_{\alpha,J}(-r)$ for all J of cardinality $r+1$ such that $e_j\alpha$ for every $j \in J$ is an integer. The filtered $\mathscr{D}_{Y^J}$-modules $\mathcal{V}_{\alpha,J}$ comes from cyclic coverings so that $\mathcal{P}_{\alpha,r}$ carries the Hodge theory of the cyclic coverings. In fact, by a well-know construction in [EV92, §3] the direct image of the de Rham complex of a cyclic covering decomposes into log de Rham complexes of line bundles. A line bundle with an integrable log connection also can be viewed as a log $\mathscr{D}$-module. This suggests that the $\mathscr{D}$-modules $\mathcal{V}_{\alpha,J}$ is generated by a certain log $\mathscr{D}$-module $\mathscr{V}_{\alpha,J}$. If Y is reduced and $\alpha = 0$, $\mathcal{V}_{\alpha,J}$ is just ω_{Y^J}. We shall construct auxiliary log $\mathscr{D}$-modules $\mathscr{V}_{\alpha,J}$ whose log de Rham complex will be used to construct the $\mathscr{D}$-module $\mathcal{V}_{\alpha,J}$, without using cyclic cover. The cyclic coverings are involved only when we study the Hodge theory of those $\mathscr{D}$-modules. We fix a rational number $\alpha \in [0,1)$ to simplify the notations and let I_α be a subset of indices consisting of i such that αe_i is an integer.

Denote by $\mathcal{L}$ the line bundle $\mathscr{O}_X\left(-\sum_{i\in I_\alpha} \frac{e_i}{N} Y_i\right)$, where N is the greatest common divisor of e_i for $i \in I_\alpha$. In this notation, $\mathscr{O}_X(-\lceil \alpha Y \rceil) = \mathcal{L}^{\alpha N}\left(-\sum_{i \in I\setminus I_\alpha} \lceil \alpha e_i Y_i \rceil\right)$. Because the line bundle $\mathscr{O}_X(Y)$ can be trivialized by a global section, we get an isomorphism of $\mathscr{O}_X$-modules:

$$\mathcal{L}^N = \mathscr{O}_X\left(-\sum_{i\in I_\alpha} e_i Y_i\right) \to \mathscr{O}_X\left(\sum_{i \in I\setminus I_\alpha} e_i Y_i\right). \tag{7.33}$$

Choose a local section l of $\mathcal{L}$ such that $l^N \mapsto \prod_{i\in I\setminus I_\alpha} z_i^{-e_i}$ under (7.33). Now we shall put a log connection ∇ on

$$\mathscr{O}_X(-\lceil \alpha Y \rceil) = \mathcal{L}^{\alpha N}\left(-\sum_{i\in I\setminus I_\alpha} \lceil \alpha e_i Y_i \rceil\right).$$

First we define, using the product rule

$$\frac{\nabla l^N}{l^N} = N\frac{\nabla l}{l} = \sum_{i\in I\smallsetminus I_\alpha} -e_i \frac{dz_i}{z_i} \tag{7.34}$$

due to (7.33). Then, let $s = l^{\alpha N}\prod_{i\in I\smallsetminus I_\alpha} z_i^{\lceil \alpha e_i\rceil}$ be the local frame of $\mathscr{O}_X(-\lceil\alpha Y\rceil)$. Noting that αN is a non-negative integer, the induced log connection works as

$$\begin{aligned}\frac{\nabla s}{s} = \frac{\nabla(l^{\alpha N}\prod_{i\in I\smallsetminus I_\alpha} z_i^{\lceil\alpha e_i\rceil})}{l^{\alpha N}\prod_{i\in I\smallsetminus I_\alpha} z_i^{\lceil\alpha e_i\rceil}} &= \alpha N\frac{\nabla l}{l} + \sum_{i\in I\smallsetminus I_\alpha}\lceil\alpha e_i\rceil\frac{dz_i}{z_i}\\ &= \sum_{i\in I\smallsetminus I_\alpha}(\lceil\alpha e_i\rceil - \alpha e_i)\frac{dz_i}{z_i} = \sum_{i\in I\smallsetminus I_\alpha}\{-\alpha e_i\}\frac{dz_i}{z_i},\end{aligned} \tag{7.35}$$

where $\{-\}$ denotes the function of taking fractional part. Putting in more standard form,

$$\nabla s = \sum_{i\in I\smallsetminus I_\alpha}\{-\alpha e_i\}\frac{dz_i}{z_i}\otimes s.$$

This log connection is integrable and has poles along Y_i for $i\in I\smallsetminus I_\alpha$ with eigenvalues $\{-\alpha e_i\}$. We endow the line bundle $\mathscr{O}_X(-\lceil\alpha Y\rceil)$ with this integrable log connection ∇.

Fix a subset J of I_α with $\#J = r+1$ so that $\dim Y^J = n-r$. The pullback of $(\mathscr{O}_X(\lceil-\alpha Y\rceil),\nabla)$ by the inclusion $\tau^J : Y^J\to X$ gives an integrable log connection $(\mathscr{V},\nabla) = (\mathscr{V}_{\alpha,J},\nabla)$ on Y^J with poles along $E = E^{\alpha,J}$ the pullback of $Y_{I\smallsetminus I_\alpha}$. Moreover, the log de Rham complex of $(\mathscr{V},\nabla)$

$$\{\mathscr{V}\to\Omega_{Y^J}(\log E)\otimes\mathscr{V}\to\cdots\to\Omega^{n-r}_{Y^J}(\log E)\otimes\mathscr{V}\}[n-r],$$

induces a complex of $\mathscr{D}_{Y^J}$-modules

$$\{\mathscr{V}\otimes\mathscr{D}_{Y^J}\to\Omega_{Y^J}(\log E)\otimes\mathscr{V}\otimes\mathscr{D}_{Y^J}\to\cdots\to\Omega^{n-r}_{Y^J}(\log E)\otimes\mathscr{V}\otimes\mathscr{D}_{Y^J}\}[n-r], \tag{7.36}$$

which is nothing but the log de Rham complex of $\mathscr{V}\otimes\mathscr{D}_{Y^J}$. It follows from Lemma 2.3 that the complex is a resolution of

$$\mathcal{V} = \mathcal{V}_{\alpha,J} =_{\mathrm{def}} \omega_{Y^J}(\log E)\otimes\mathscr{V}\underset{\mathscr{D}_{(Y^J,E)}}{\otimes}\mathscr{D}_{Y^J}.$$

We endow $\mathcal{V}$ with the filtration $F_\ell\mathcal{V} = F_\ell\mathcal{V}_{\alpha,J}$ induced the subcomplex

$$\{\mathscr{V}\otimes F_\ell\mathscr{D}_{Y^J}\to\Omega_{Y^J}(\log E)\otimes\mathscr{V}\otimes F_{\ell+1}\mathscr{D}_{Y^J}\to\cdots\to\Omega^{n-r}_{Y^J}(\log E)\otimes\mathscr{V}\otimes F_{\ell+n-r}\mathscr{D}_{Y^J}\}[n-r].$$

It is clear that $F_\bullet\mathcal{V}$ is a good filtration. For example, if $\alpha = 0$, then E is empty and $\mathscr{V}$ is just $\mathscr{O}_{Y^J}$ so that $\mathcal{V} = \omega_{Y^J}$ as $\mathscr{D}_{Y^J}$-modules. Since the eigenvalues of the log connection are in $(0,1)$ if poles exist, the log de Rham complex of $(\mathscr{V},\nabla)$ is the minimal extension $R_{!*}\mathbb{V}$ of the local system $\mathbb{V}$ consisting of the flat sections of ∇ on $\mathscr{V}$ over $Y^J\smallsetminus Y_{I\smallsetminus J}$ (see [EV92, 1.6]). Later we will put a sesquilinear pairing on $\mathcal{V}$ and all the data will yield a pure Hodge structure of the log de Rham complex of $\mathscr{V}$.

Lemma 7.9. *The de Rham complex* $\mathrm{DR}_{Y^J}\mathcal{V}$ *together with the filtration* $F_\bullet\mathrm{DR}_{Y^J}\mathcal{V}$ *is isomorphic to the log de Rham complex* $\Omega^{n-r+\bullet}_{Y^J}(\log E)\otimes\mathscr{V}$ *with the stupid filtration in the derived category of filtered complexes of* $\mathbb{C}$*-vector spaces. In addition,* $\mathcal{V}$ *is holonomic and the characteristic cycle of* $\mathcal{V}$ *is*

$$cc(\mathcal{V}) = \sum_{K\subset I\smallsetminus I_\alpha}\left[T^*_{Y^{K\cup J}}Y^J\right].$$

Proof. We can choose the local frame s of $\mathscr{V}$ such that

$$\nabla s = \sum_{i \in I \setminus I_\alpha} \frac{dz_i}{z_i} \otimes \{-\alpha e_i\} s$$

where z_i is the defining equation of E_i for each i. Therefore, the complex (7.36) locally is the Koszul complex over $\mathscr{D}_{Y^J}$ associated to the sequence

$$x_1\partial_1 + \{-\alpha e_1\}, x_2\partial_2 + \{-\alpha e_2\}, ..., x_p\partial_p + \{-\alpha e_p\}, \partial_{p+1}, \partial_{p+2}, ..., \partial_{n-r},$$

for some rearrangement of coordinates and under the trivialization of $\mathscr{V}$ given by s. It follows that the associated graded of (7.36) is the Koszul complex associated to the regular sequence

$$x_1\partial_1, x_2\partial_2, ..., x_p\partial_\nu, \partial_{p+1}, \partial_{p+2}, ..., \partial_{n-r}$$

over $\mathrm{gr}^F \mathscr{D}_{Y^J}$. Thus, the complex (7.36) is filtered acyclic. By the similar argument in Theorem 4.5, the $\mathscr{D}_{Y^J}$-module $\mathcal{V}$ is holonomic and the charateristic cycle $cc(\mathcal{V}) = \sum_{K \subset I \setminus I_\alpha} \left[T^*_{Y^{K \cup J}} Y^J\right]$.

Moreover, we have isomorphisms in the derived category of complexes of $\mathbb{C}$-vector spaces:

$$\begin{aligned} F_\ell \mathrm{DR}\mathcal{V} = F_{\ell+*}\mathcal{V} \otimes \bigwedge^{-\star} \mathscr{T}_{Y^J} &\simeq \Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r+\bullet+*}\mathscr{D}_{Y^J} \otimes \bigwedge^{-\star} \mathscr{T}_{Y^J} \\ &\simeq \Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r}\mathscr{O}_{Y^J}. \end{aligned}$$

Since $F_\ell \mathscr{O}_{Y^J}$ is $\mathscr{O}_{Y^J}$ or vanishes if $\ell < 0$, the complex $\Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r}\mathscr{O}_{Y^J}$ is the stupid filtration on the log de Rham complex on $\mathscr{V}$. We conclude the proof. □

We also need an auxiliary $\mathscr{D}_{Y^J}$-module $\mathcal{V}^*_{\alpha,J}$ to identify the primitive part $\mathcal{P}_{\alpha,r}$ which plays the role as $\omega_{Y^J}(*D^J)$ in the counterpart for the reduced case (Theorem 5.7). The log de Rham complex of $(\mathscr{V}, \nabla)$ can be enlarged into

$$\{\mathscr{V} \to \Omega_{Y^J}(\log D) \otimes \mathscr{V} \to \cdots \to \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V}\}[n-r], \quad \text{for } D = D^J \text{ the pullback of the divisor } Y_{I \setminus J}.$$

It is quasi-isomorphic to $Rj_*\mathbb{V}$ for $j : Y^J \setminus Y_{I_\alpha} \to Y^J$ is the open immersion. By the similar process of the above, it induces a filtered acyclic complex of $\mathscr{D}_{Y^J}$-modules

$$\{\mathscr{V} \otimes \mathscr{D}_{Y^J} \to \Omega_{Y^J}(\log D) \otimes \mathscr{V} \otimes \mathscr{D}_{Y^J} \to \cdots \to \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V} \otimes \mathscr{D}_{Y^J}\}[n-r]. \tag{7.37}$$

Let $\mathcal{V}^* = \mathcal{V}^*_{\alpha,J}$ be the 0-th cohomology of the above complex and endow it with the filtration such that $F_\ell \mathcal{V}^* = F_\ell \mathcal{V}^*_{\alpha,J}$ is induced by the subcomplex

$$\{\mathscr{V} \otimes F_\ell \mathscr{D}_{Y^J} \to \Omega_{Y^J}(\log D) \otimes \mathscr{V} \otimes F_{\ell+1}\mathscr{D}_{Y^J} \to \cdots \to \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V} \otimes F_{\ell+n-r}\mathscr{D}_{Y^J}\}[n-r].$$

We naturally get an induced morphism $(\mathcal{V}, F_\bullet\mathcal{V}) \to (\mathcal{V}^*, F_\bullet\mathcal{V}^*)$ from the inclusion of the log de Rham complexes.

Lemma 7.10. *The canonical morphism $(\mathcal{V}, F_\bullet\mathcal{V}) \to (\mathcal{V}^*, F_\bullet\mathcal{V}^*)$ is injective, whose image is generated by the monomials defining $D - E$.*

Proof. Suppose $x_1x_2\cdots x_p$ is the local defining equation of E and $x_1x_2\cdots x_q$ is the local defining equation of D for $q \geq p+1$. Since $\mathcal{V}$ is locally generated by the class of

$$\bigwedge_{i=1}^{p} \frac{dx_i}{x_i} \wedge dx_{p+1} \wedge \cdots \wedge dx_{n-r} \otimes s \otimes 1$$

and $\mathcal{V}^*$ is locally generated by the class of

$$\bigwedge_{i=1}^{q} \frac{dx_i}{x_i} \wedge dx_{q+1} \wedge \cdots \wedge dx_{n-r} \otimes s \otimes 1,$$

the image is generated by the class of $\bigwedge_{i=1}^{q}\frac{dx_i}{x_i}\wedge dx_{q+1}\wedge\cdots\wedge dx_{n-r}\otimes s\otimes x_{p+1}x_{p+2}\cdots x_q$. The morphism locally is

$$\mathscr{D}_{Y^J}/(x_1\partial_1+r_1,...,x_p\partial_p+r_p,\partial_{p+1},...,\partial_{n-r})\mathscr{D}_{Y^J}\to\mathscr{D}_{Y^J}/(x_1\partial_1+r_1,...,x_q\partial_q+r_q,\partial_{q+1}...,\partial_{n-r})\mathscr{D}_{Y^J},$$

with $[P]\mapsto[x_{p+1}x_{p+2}\cdots x_qP]$ where $r_1,r_2,...,r_p$ are the eigenvalues of ∇ on $\mathscr{V}$ and $r_{p+1}=r_{p+2}=\cdots=r_q=0$. Since

$$\Omega^{n-r}_{Y^J}(\log E)\otimes\mathscr{V}=F_{-(n-r)}\mathcal{V}\to F_{-(n-r)}\mathcal{V}^*=\Omega^{n-r}_{Y^J}(\log D)\otimes\mathscr{V}$$

is injective, by induction, it suffices to show that $\mathrm{gr}^F\mathcal{V}\to\mathrm{gr}^F\mathcal{V}^*$ is injective. Due to the complexes (7.36) and (7.37) is filtered acyclic, the morphism on the associated graded modules works as, in the local representation,

$$\mathrm{gr}^F\mathscr{D}_{Y^J}/(x_1\partial_1,...,x_p\partial_p,\partial_{p+1},...,\partial_{n-r})\mathrm{gr}^F\mathscr{D}_{Y^J}\to\mathrm{gr}^F\mathscr{D}_{Y^J}/(x_1\partial_1,...,x_q\partial_q,\partial_{q+1}...,\partial_{n-r})\mathrm{gr}^F\mathscr{D}_{Y^J},$$

with $[P]\mapsto[x_{p+1}x_{p+2}\cdots x_qP]$. By induction on the number of components of $D-E$, we can assume $q=p+1$. Let $P\in\mathrm{gr}^F\mathscr{D}_{Y^J}$ represent a class in the kernel. Then

$$x_qP=\sum_{i=1}^{q}x_i\partial_iP_i+\sum_{j=q+1}^{n-r}\partial_jP_j\in\mathrm{gr}^F\mathscr{D}_{Y^J}.$$

Subtracting $x_q\partial_qP_q$ on the both sides yeilds

$$x_q(P-\partial_qP_q)=\sum_{i=1}^{q-1}x_i\partial_iP_i+\sum_{j=q+1}^{n-r}\partial_jP_j\in\mathrm{gr}^F\mathscr{D}_{Y^J}.$$

Since $x_q,x_1\partial_1,...,x_{q-1}\partial_{q-1},\partial_{q+1},...,\partial_{n-r}$ is a regular sequence over $\mathrm{gr}^F\mathscr{D}_{Y^J}$,

$$(P-\partial_qP_q)=\sum_{i=1}^{q-1}x_i\partial_iP'_i+\sum_{j=q+1}^{n-r}\partial_jP'_j\in\mathrm{gr}^F\mathscr{D}_{Y^J}.$$

We find that P is a linear combination of $x_1\partial_1,x_2\partial_2,...,x_p\partial_p,\partial_{p+1},...,\partial_{n-r}$ over $\mathrm{gr}^F\mathscr{D}_{Y^J}$. We conclude the proof. □

Remark 7.11. One can use Riemann-Hilbert correspondence to conclude that $\mathcal{V}$ is the minimal extension of $\mathscr{V}|_{Y^J\smallsetminus D}$ and $\mathcal{V}^*$ is the $*$-extension of $\mathscr{V}|_{Y^J\smallsetminus D}$, which is overkill in our situation. The above argument also showed the strictness, i.e., $F_\ell\mathcal{V}=F_\ell\mathcal{V}^*\cap\mathcal{V}$.

Putting in more general notations and summarizing what we have proved in the above two lemmas:

Theorem 7.12. *The filtered $\mathscr{D}_{Y^J}$-module $(\mathcal{V}_{\alpha,J},F_\bullet)$ is holonomic whose de Rham complex $\mathrm{DR}_{Y^J}\mathcal{V}_{\alpha,J}$ together with the induced filtration is isomorphic to the log de Rham complex $\Omega^{n-r+\bullet}_{Y^J}(\log E^{\alpha,J})\otimes\mathscr{V}_{\alpha,J}$ with the stupid filtration in the derived category of filtered complexes of $\mathbb{C}$-vector spaces and whose characteristic cycle is*

$$cc(\mathcal{V}_{\alpha,J})=\sum_{K\subset I\smallsetminus I_\alpha}\left[T^*_{Y^{K\cup J}}Y^J\right].$$

The canonical filtered morphism $(\mathcal{V}_{\alpha,J},F_\bullet\mathcal{V}_{\alpha,J})\to(\mathcal{V}^*_{\alpha,J},F_\bullet\mathcal{V}^*_{\alpha,J})$ is injective and the image is generated by the monomial defining the divisor $D^J-E^{\alpha,J}$.

7.5. **Identifying the primitive part $\mathcal{P}_{\alpha,r}$.** Now we are going to identify the r-th primitive part $(\mathcal{P}_{\alpha,r},F_\bullet\mathcal{P}_{\alpha,r})$ with a direct sum of $\mathcal{V}_{\alpha,J}(-r)$ for J ranging over subsets I_α of cardinality $r+1$. The argument is parallel to the one of the reduced case (Theorem 5.7), replacing $\mathcal{M}$ by $\mathcal{M}_\alpha$, R by R_α, ω_{Y^J} by $\mathcal{V}_{\alpha,J}$, $\omega_{Y^J}(*D^J)$ by $\mathcal{V}^*_{\alpha,J}$, the complex $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y$ by $C^\bullet_\alpha=\Omega^{n+\bullet}_{X/\Delta}(\log Y)(-\lceil\alpha Y\rceil)|_{Y_{I_\alpha}}$ and the log de Rham complex $\Omega^{n-r+\bullet}_{Y^J}(\log D^J)$ by $\Omega^{n-r+\bullet}_{Y^J}(\log D^J)\otimes\mathscr{V}_{\alpha,J}$.

Theorem 7.13. *Let $\mathcal{V}_{\alpha,r}=\bigoplus_J\tau^J_+\mathcal{V}_{\alpha,J}$ for J running over the subsets of I_α of cardinality $r+1$, where $\tau^J:Y^J\hookrightarrow X$ is the closed embedding. Then there exists an isomorphism $\phi_{\alpha,r}:(\mathcal{P}_{\alpha,r},F_\bullet\mathcal{P}_{\alpha,r})\to\mathcal{V}_{\alpha,r}(-r)$ in the category of filtered $\mathscr{D}_X$-modules.*

Proof. Because the log connection (7.33) we constructed on $\mathscr{O}_X(-\lceil \alpha Y \rceil)$ has zero residue on Y_i for $i \in I_\alpha$, we have the residue morphism between log de Rham complexes.

$$\mathrm{Res}_{Y^J} : \Omega_X^{\bullet+n+1}(\log Y) \otimes \mathscr{O}_X(-\lceil \alpha Y \rceil)|_{Y_{I_\alpha}} \to \Omega_{Y^J}^{\bullet+n-r}(\log D^J) \otimes \mathscr{V}_{\alpha,J}, \text{ where } D^J \text{ is the pull back of } Y_{I \smallsetminus J}$$

for $J \subset I_\alpha$ of cardinality $r+1$, up to a sign depending on the order of the indices. Denote by $B_\alpha^\bullet$ the log de Rham complex $\Omega_X^{\bullet+n+1}(\log Y) \otimes \mathscr{O}_X(-\lceil \alpha Y \rceil)$ of $\mathscr{O}_X(-\lceil \alpha Y \rceil)$. The residue morphism Res_{Y^J} extends to a morphism of the complexes of induced $\mathscr{D}_X$-modules

$$\mathrm{Res}_{Y^J} : B_\alpha^\bullet|_{Y_{I_\alpha}} \otimes \mathscr{D}_X \to \Omega_{Y^J}^{\bullet+n-r}(\log D^J) \otimes \mathscr{V}_{\alpha,J} \otimes \mathscr{D}_X.$$

Let $\mathcal{H}_\alpha^\ell$ be the ℓ-th cohomology of $B_\alpha^\bullet|_{Y_{I_\alpha}} \otimes \mathscr{D}_X$. Then we have an induced morphism $\mathrm{Res}_{Y^J} : \mathcal{H}_\alpha^0 \to \mathcal{V}_{\alpha,J}^*$ by taking cohomology. Let $\mathrm{Res}_{\alpha,r} = \bigoplus \mathrm{Res}_{Y^J} : \mathcal{H}_\alpha^0 \to \mathcal{V}_{\alpha,r}^*(-r)$ where $\mathcal{V}_{\alpha,r}^* = \bigoplus_J \mathcal{V}_{\alpha,J}^*$ for J running over cardinality $r+1$ subsets of I_α. Because $\frac{dt}{t}\wedge : \Omega_{X/\Delta}^{\bullet+n}(\log Y)(-\lceil \alpha Y \rceil) \to \Omega_X^{\bullet+n+1}(\log Y)(-\lceil \alpha Y \rceil)$ also extends to the complexes of the induced $\mathscr{D}_X$-modules, we obtain a short exact sequence

$$0 \longrightarrow C_\alpha^{\bullet-1} \otimes \mathscr{D}_X \xrightarrow{\frac{dt}{t}\wedge} B^{\bullet-1}|_{Y_{I_\alpha}} \otimes \mathscr{D}_X \longrightarrow C_\alpha^\bullet \otimes \mathscr{D}_X \longrightarrow 0.$$

The associated long exact sequence gives

$$0 \longrightarrow \mathcal{H}_\alpha^{-1} \longrightarrow \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\alpha \xrightarrow{\frac{dt}{t}\wedge} \mathcal{H}_\alpha^0 \xrightarrow{R_\alpha} 0. \tag{7.38}$$

By pre-composing $\frac{dt}{t}\wedge$, we get a morphism

$$\mathrm{Res}_{\alpha,J} \circ \frac{dt}{t}\wedge : \mathcal{M}_\alpha \to \mathcal{V}_{\alpha,r}^*(-r), \quad [\zeta_\alpha \otimes P] \to [\mathrm{Res}_{\alpha,J} \frac{dt}{t} \wedge \zeta_\alpha \otimes P].$$

Recall that every element in $\mathcal{M}_\alpha$ is locally represented by $\zeta_\alpha \otimes P$ for $\zeta_\alpha = z_I^{\lceil \alpha \mathbf{e} \rceil} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ given that locally $I = \{0, 1, \ldots, k\}$, and $P \in \mathscr{D}_X$. By Corollary 7.7, every class in $\ker R_\alpha^{r+1}$ is represented by $\zeta_\alpha \otimes z_{\overline{J}} P$ for some ordered index subset J of I_α of cardinality $r+1$ and $\overline{J}$ is the complement of J in I_α and $z_{\overline{J}} = \prod_{j \in \overline{J}} z_j$. Thus, its image under the above morphism only contained in the component $\mathcal{V}_{\alpha,J}^*(-r)$ because $z_{\overline{J}}$ vanishes on other components. The image is the class represented by

$$\mathrm{Res}_{\alpha,J} \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n z_I^{\lceil \alpha \mathbf{e} \rceil} \otimes z_{\overline{J}} P = \pm \frac{dz_{I \smallsetminus J}}{z_{I \smallsetminus J}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes z_{\overline{J}} P \in \Omega_{Y^J}^{n-r} \otimes \mathscr{V}_{\alpha,J} \otimes \mathscr{D}_X, \tag{7.39}$$

where $s_{\alpha,J}$ is the local frame of $\mathscr{V}_{\alpha,J}$ by restricting $z_I^{\lceil \alpha \mathbf{e} \rceil}$ and the sign is depending on the order of J. It also follows from the calculation that the image does not have pole along the pull-back of $Y_{\overline{J}}$. So it is contained in the subsheaf consisting of classes represented by $\Omega_{Y^J}^{n-r}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J} \otimes \mathscr{D}_X$, where $E^{\alpha,J}$ is the pull-back of $Y_{I \smallsetminus I_\alpha}$ so that $D^J - E^{\alpha,J}$ is the pull-back of $Y_{\overline{J}}$. This means that the image of the class represented by (7.39) is also in the image of the canonical inclusion:

$$\tau_+^J \mathcal{V}_{\alpha,J}(-r) \hookrightarrow \tau_+^J \mathcal{V}_{\alpha,J}^*(-r),$$
$$[dz_{\overline{J}} \wedge \frac{dz_{I \smallsetminus I_\alpha}}{z_{I \smallsetminus I_\alpha}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes P] \mapsto [\frac{dz_{\overline{J}}}{z_{\overline{J}}} \wedge \frac{dz_{I \smallsetminus I_\alpha}}{z_{I \smallsetminus I_\alpha}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes z_{\overline{J}} P].$$

See Theorem 7.12. Therefore, the morphism $\ker R_\alpha^{r+1} \to \mathcal{V}^*_{\alpha,r}(-r)$ constructed above factors through $\mathcal{V}_{\alpha,r}(-r)$. Summarizing, we have the following diagram.

$$\begin{array}{ccccc} \ker R_\alpha^{r+1} & & \xrightarrow{\rho_{\alpha,r}} & & \mathcal{V}_{\alpha,r}(-r) \\ \downarrow & & & & \downarrow \\ \mathcal{M}_\alpha & \xrightarrow{\frac{dt}{t}\wedge} & \mathcal{H}^0_\alpha & \xrightarrow{\mathrm{Res}_{\alpha,r}} & \mathcal{V}^*_{\alpha,r}(-r) \end{array}$$

In fact, the kernel of ρ_r contains $\ker R_\alpha^r$: for an element in $\ker R_\alpha^r$ locally represented by $\zeta_\alpha \otimes z_K P$ for K a subset of I_α such that the cardinality of $I_\alpha \smallsetminus K$ is r, its image under $\rho_{\alpha,r}$ is zero because z_K annihilates all $\Omega^{n-r}_{Y^J}(\log D^J)\otimes\mathscr{V}_{\alpha,J}$ for any $J \subset I_\alpha$ of cardinality $r+1$. The morphism $\rho_{\alpha,r}$ also kills $R_\alpha \ker R_\alpha^{r+2}$ because $\frac{dt}{t}\wedge$ vanishes on the image of R_α by (7.38). It follows that $\rho_{\alpha,r}$ factors through a filtered morphism

$$\phi_{\alpha,r} : \mathcal{P}_{\alpha,r} = \frac{\ker R_\alpha^{r+1}}{\ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}} \to \mathcal{V}_{\alpha,r}(-r).$$

For $dz_{\overline{J}} \wedge \frac{dz_{I\smallsetminus I_\alpha}}{z_{I\smallsetminus I_\alpha}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes P \in \Omega^{n-r}_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J} \otimes F_\ell \mathscr{D}_X$ representing a class in $F_\ell \tau_+^J \mathcal{V}_{\alpha,J}(-r)$ where $J \subset I_\alpha$ of cardinality $r+1$, we can find a lifting represented by $\zeta_\alpha \otimes z_{\overline{J}} P$ in $F_\ell \ker R_\alpha^{r+1}$, which means

$$F_\ell \ker R_\alpha^{r+1} \to F_{\ell+r} \mathcal{V}_{\alpha,r}$$

is surjective, i.e. the morphism $\phi_{\alpha,r}$ is filtered surjective. It remains to prove that $\phi_{\alpha,r}$ is injective. We prove that $\phi_{\alpha,r}$ is an isomorphism by counting the characteristic cycles as in Theorem 5.7. Because $\phi_{\alpha,r}$ is surjective, one gets

$$cc(\mathcal{P}_{\alpha,r}) \geq cc(\mathcal{V}_{\alpha,r}).$$

It follows from Corollary 7.12 that

$$cc(\mathcal{V}_{\alpha,r}) = \sum_{\substack{J\subset I_\alpha,\\ \#J=r+1}} cc(\tau_+^J \mathcal{V}_{\alpha,J}) = \sum_{\substack{J\subset I_\alpha,\\ \#J=r+1}} \sum_{K\subset I\smallsetminus I_\alpha} [T^*_{Y^{J\cup K}} X] = \sum_{\substack{J\subset I,\\ \#J\cap I_\alpha=r+1}} [T^*_{Y^J} X].$$

One the other hand, by the Lefschetz decomposition and Theorem 7.2,

$$\begin{aligned} \sum_{J\subset I} \#(J\cap I_\alpha)\,[T^*_{Y^J}X] = cc(\mathcal{M}_\alpha) = cc(\mathrm{gr}^W \mathcal{M}_\alpha) &= \sum_{r\geq 0}(r+1)cc(\mathcal{P}_{\alpha,r}) \geq \sum_{r\geq 0}(r+1)cc(\mathcal{V}_{\alpha,r}) \\ &= \sum_{r\geq 0} \sum_{\substack{J\subset I,\\ \#J\cap I_\alpha=r+1}} (r+1)\,[T^*_{Y^J}X] = \sum_{J\subset I} \#(J\cap I_\alpha)\,[T^*_{Y^J}X]. \end{aligned}$$

It follows that all inequalities above are equalities and in particular,

$$cc(\mathcal{P}_{\alpha,r}) = cc(\mathcal{V}_{\alpha,r})$$

from which we conclude that $\phi_{\alpha,r}$ is an isomorphism between the underlying $\mathscr{D}_X$-modules. Plus it is filtered surjective, we conclude that $\phi_{\alpha,r}$ is filtered isomorphism. □

8. Non-reduced case: Sesquilinear pairing and limiting mixed Hodge structure

8.1. **Kähler package of cyclic covering.** To accomplish our goal, we need to show that the sum of all hypercohomologies of the complex

$$\Omega^\bullet_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J}[n-r]$$

has a polarized Hodge-Lefschetz structure and hard Lefschetz so that the hypercohomology of the de Rham complex of the primitive part $\mathcal{P}_{\alpha,r}$ will inherit the properties by Theorem 7.12 and Theorem 7.13. For this, we need to use the geometry of cyclic coverings.

We first give another description of the integrable log connection (7.33) using cyclic coverings. Fix a rational number α in $[0,1)$, Because the isomorphism,

$$\mathcal{L}^N = \mathscr{O}_X\left(-\sum_{i\in I_\alpha} e_iY_i\right) \to \mathscr{O}_X\left(\sum_{i\in I\smallsetminus I_\alpha} e_iY_i\right),$$

we obtain a cyclic covering $\pi_\alpha : X_\alpha \to X$ by taking the N-th roots out of $\sum_{i\in I\smallsetminus I_\alpha} e_iY_i$ and normalizing it. The direct image $\pi_{\alpha*}\mathscr{O}_{X_\alpha}$ decomposes into eigenspaces with respect the Galois action as well as the direct image of exterior differential $\pi_{\alpha*}\mathscr{O}_{X_\alpha} \to \pi_{\alpha*}\Omega_{X_\alpha}$ [EV92, Theorem 3.2]. The line bundle

$$\mathcal{L}^{\alpha N}\left(-\sum_{i\in I\smallsetminus I_\alpha} \lceil \alpha e_iY_i\rceil\right),$$

is the α-eigenspaces of $\pi_{\alpha*}\mathscr{O}_{X_\alpha}$ for some suitable choice of a generator of the Galois group. Because the decomposition respects the exterior differential, we obtained an integrable log connection with eigenvalues $\{\alpha e_i\}$ along Y_i for each $i\in I_\alpha$. Note that X_α might not be smooth.

Let $J\subset I_\alpha$ of cardinality $r+1$. Since Y^J is not contained in $Y_{I\smallsetminus I_\alpha}$, the fiber product $Y^J_\alpha = X_\alpha\times_X Y^J$ is again a cyclic covering of Y^J by taking the N-th roots out of $\sum_{i\in I\smallsetminus I_\alpha} e_iY_i\cap Y^J$. Let $\pi^J_\alpha : Y^J_\alpha \to Y^J$ be the second projection.

$$\begin{array}{ccc} Y^J_\alpha & \longrightarrow & X_\alpha \\ \big\downarrow{\scriptstyle \pi^J_\alpha} & & \big\downarrow{\scriptstyle \pi_\alpha} \\ Y^J & \xrightarrow{\tau^J} & X \end{array} \tag{8.40}$$

We conclude that $(\mathscr{V}_{\alpha,J},\nabla)$ is the α-eigenspace of $\pi^J_{\alpha*}(\mathscr{O}_{Y^J_\alpha},d)$. The log de Rham complex of $(\mathscr{V}_{\alpha,J},\nabla)$ is a summand of the direct image of the de Rham compolex $\pi^J_{\alpha*}\Omega^{\bullet+n-r}_{Y^J_\alpha}$ of Y^J_α.

We shall work in the general setting and adopt the convention in [EV86] and [EV92]. Let $\mathcal{L}$ be a line bundle on a Kähler manifold Z with a Kähler form ω and $D=\sum_i \nu_iD_i$ be a simple normal crossings divisor such that for some $N>1$ one has $\mathcal{L}^N = \mathscr{O}_Z(D)$. Define $\mathcal{L}^{(j)} = \mathcal{L}^j(-\lfloor \frac{jD}{N}\rfloor)$ for $1\le j\le N-1$. One puts an integrable logarithmic connection on $\mathcal{L}^{(j)}$ with poles along $D^{(j)}$, where

$$D^{(j)} = \sum_{\frac{j\nu_i}{N}\notin\mathbb{Z}} D_i.$$

Let $\iota: U\hookrightarrow Z$ be the complement of D and $\mathbb{V}$ is the underlying local system of $\mathcal{L}|_U$. Let $\tau: Z'\to Z$ be the cyclic covering obtained by first taking N-th root out of D then taking the normalization and $\pi:\tilde{Z}\to Z'$ be a log resolution of singularity equivariant with respect to the Galois group $\mathrm{Gal}\,(Z'/Z) = \langle\sigma\rangle$ and let E be the simple normal crossing exceptional divisor.

$$\tilde{Z} \xrightarrow{\pi} Z' \xrightarrow{\tau} Z, \qquad \eta = \tau\circ\pi$$

Note that $\tilde{Z}$ is Kähler because it is a resolution of subvariety of the geometric line bundle of $\mathcal{L}$, which is Kähler, although the induced Kähler class does not relate well with ω on X. The pullback $\eta^*\omega$ is only positive over $\tilde{U} = \eta^{-1}(U)$, but one can still cook up a Kähler class by adding a small multiple of the first Chern class $\Theta\in H^2(\tilde{Z},\mathbb{Z}(1))$ of the relative ample line bundle of the projective morphism $\pi:\tilde{Z}\to Z'$. We can assume Θ is invariant under σ by averaging it.

Lemma 8.1. *Notations as above, the cohomology class* $[\eta^*\omega]+\lambda(2\pi\sqrt{-1})^{-1}\Theta\in H^{1,1}(Z)\cap H^2(Z,\mathbb{R})$ *is an invariant Kähler class for* λ *is a sufficient small positive number.*

Proof. Let $\tilde{D}_i$ be the strict transformation of $\tau^{-1}(D_i)$ and $s_i \in H^0(\tilde{Z}, \mathscr{O}_{\tilde{Z}}(\tilde{D}_i))$ whose zero locus is $\tilde{D}_i$. Let h_i be a Hermitian metric on each line bundle $\mathscr{O}_{\tilde{Z}}(\tilde{D}_i)$ and ρ_i be sufficient small positive bump function supported in a small neighborhood of $\tilde{D}_i$ for each i. Then the $(1,1)$-form

$$\eta^*\omega + \sum_i \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho_i h_i(s_i, s_i)$$

is positive on $\tilde{Z} - E$ but only semi-positive over E. However, the class $(2\pi\sqrt{-1})^{-1}\Theta$ is positive over E. Therefore, for λ sufficient small positive, the class of

$$\eta^*\omega + \sum_i \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho_i h_i(s_i, s_i) + \lambda(2\pi\sqrt{-1})^{-1}\Theta$$

is a Kähler class. But $\partial\bar{\partial}\rho_i h_i(s_i, s_i)$ is exact. The cohomology class of above just equals $[\eta^*\omega] + \lambda(2\pi\sqrt{-1})^{-1}\Theta$ in $H^{1,1}(\tilde{Z}) \cap H^2(Z, \mathbb{R})$. It is invariant because both $[\eta^*\omega]$ and Θ are invariant. □

Lemma 8.2. *The hypercohomology $H^k\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$ is a summand of ξ^{-j}-eigenspace of $H^k(\tilde{Z})$, and thus it is a sub-Hodge structure of weight k.*

Proof. It follows from (1.6) in [EV86] that $R\iota_!\mathbb{V}^{-j}$, $R\iota_*\mathbb{V}^{-j}$ and $\Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}$ are all quasi-isomorphic. Taking hypercohomology gives canonical isomorphisms

$$H^k(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}) \simeq H_c^k(U, \mathbb{V}^{-j}) \simeq H^k(U, \mathbb{V}^{-j}).$$

Because η is étale over U, $H^k(U, \mathbb{V}^j)$ (resp. $H_c^k(U, \mathbb{V}^j)$) is a ξ^j-eigenspace of $H^k(\tilde{U}, \mathbb{C})$ (resp. $H_c^k(\tilde{U}, \mathbb{C})$) for the cyclic action σ, where ξ is a N-th root of unity. Then the canonical morphisms of mixed Hodge structures

$$H_c^k(\tilde{U}) \to H^k(\tilde{Z}) \to H^k(\tilde{U}) \tag{8.41}$$

respect the eigenspaces decomposition because we make $\tilde{Z}$ equivariant. We complete the proof. □

Lemma 8.3. *Let $\mathsf{X} = 2\pi\sqrt{-1}L$ where $L = [\omega]\wedge$ is the Lefschetz operator on Z. The following two statements hold:*

(1) Hard Lefschetz is valid on the hypercohomolgy, i.e.

$$\mathsf{X}^k : H^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z + k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)(k)$$

is an isomorphism of Hodge structures.

(2) The pairing

$$(m', m'') \mapsto \frac{\varepsilon(\dim Z + k + 1)}{(2\pi\sqrt{-1})^{\dim Z}} \int_{\tilde{Z}} \eta^*\left(\mathsf{X}^{\dim Z - k} m' \wedge \overline{m''}\right) \tag{8.42}$$

is a polarization on the primitive part of $H^k\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$, where $\eta^*\left(\mathsf{X}^{\dim Z - k}\alpha \wedge \overline{\beta}\right)$ is the top form determined by the inclusion $\eta^*\Omega_Z^{\dim Z}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}} \subset \omega_{\tilde{Z}}$.

Proof. Let $\tilde{L} = [\eta^*\omega + \lambda\Theta]\wedge$ be the Lefschetz operator on $\tilde{Z}$. Then the Hard Lefschetz on $\tilde{Z}$ says

$$\tilde{\mathsf{X}}^k : H^{\dim Z - k}(\tilde{Z}) \to H^{\dim Z + k}(\tilde{Z})(k)$$

is an isomorphism, where $\tilde{\mathsf{X}} =_{\rm def} 2\pi\sqrt{-1}\tilde{L}$. Because $\tilde{L}$ is invariant and respects the morphisms in (8.41), the above isomorphism is compatible with eigenspaces decomposition, it follows that

$$\tilde{\mathsf{X}}^k : H^{\dim Z-k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z+k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)(k) \tag{8.43}$$

is injective by Lemma 8.2. In fact, the ξ^i-eigenspace of $H_c^k(\tilde{U})$ is orthogonal to the ξ^j-eigenspace of $H^{2\dim Z-k}(\tilde{U})$ with respect to Poincaré pairing unless $i+j \equiv 0 \,(\mathrm{mod}\, N)$: for a in the ξ^i-eigenspace of $H_c^k(\tilde{U})$ and b in the ξ^j-eigenspace of $H^{2\dim Z-k}(\tilde{U})$ then

$$\xi^i \int_{\tilde{U}} a \wedge b = \int_{\tilde{U}} \sigma^* a \wedge b = \int_{\tilde{U}} a \wedge (\sigma^{-1})^* b = \xi^{-j} \int_{\tilde{U}} a \wedge b,$$

which means $\int_{\tilde{U}} a \wedge b$ is zero unless $i+j \equiv 0 \,(\mathrm{mod}\, N)$. It follows from Poincaré duality on $H_c^k(\tilde{U}) \times H^{2\dim Z-k}(\tilde{U})$ that the ξ^i-eigenspace of $H_c^k(\tilde{U})$ is Poincaré dual to the ξ^{-i}-eigenspace of $H^{2\dim Z-k}(\tilde{U})$. On the other hand, since the ξ^i-eigenspace is complex conjugate to the ξ^{-i}-eigenspace, the ξ^i-eigenspace of $H_c^k(\tilde{U})$ and the ξ^i-eigenspace of $H^{2\dim Z-k}(\tilde{U})$ have the same dimension. It follows that the morphism (8.43) is an isomorphism.

The operator $\tilde{L}$ has the same effect as $\eta^* L$ over $H_c^\bullet(\tilde{U})$, because Θ is supported on E. Therefore,

$$\mathsf{X}^k : H^{\dim Z-k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z+k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)(k)$$

is an isomorphism. We conclude (1). It also follows that η^* identifies the primitive part of X

$$H_{\rm prim}^{\dim Z-k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$$

with the primitive part of $\tilde{\mathsf{X}}$

$$\ker\left(\tilde{\mathsf{X}}^{k+1} : H^{\dim Z-k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z+k+2}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)\right).$$

Thus, $H_{\rm prim}^{\dim Z-k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$ is a sub-Hodge structure of $H_{\rm prim}^{\dim Z-k}(\tilde{Z})$. And the restriction of the polarization is again a polarization. This proves (2). □

The above two lemmas indicate that the sum of hypercohomologies

$$\bigoplus_{k\in\mathbb{Z}} H^k\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$$

is a polarized sub-Hodge-Lefschetz structure of $\bigoplus_{k\in\mathbb{Z}} H^k(\tilde{Z}, \mathbb{C})$. In practice, it is more convenient to make the polarization independent of the resolution of singularities and intrinsic on Z. Heuristically, the local system $\mathbb{V}^{-j}$ over U inherits a pairing from $\mathbb{C}_{\tilde{U}}$ and it has a Hodge theoretic extension on its canonical extension. First, we can resolve $\Omega_Z^\bullet(\log D^{(j)})$ using $\mathcal{A}_Z^\bullet(\log D^{(j)})$, the complex of $\mathscr{C}^\infty$-forms with log poles along $D^{(j)}$. Note that we have the inclusion of sheaves

$$\mathcal{A}_Z^{\dim Z+k}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}} \wedge \overline{\mathcal{A}_Z^{\dim Z-k}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}} \subset \mathcal{A}_Z^{2\dim Z} \otimes \mathcal{L}^{(j)^{-1}}(D^{(j)}) \underset{\mathbb{C}}{\otimes} \overline{\mathcal{L}^{(j)^{-1}}(D^{(j)})}.$$

Since $\mathcal{L}^N \simeq \mathscr{O}_Z(D)$, picking local section of l such that $l^N = \prod_i x_i^{-\nu_i}$ we can put a canonical singular Hermitian metric on $\mathcal{L}$ by setting the weight function as

$$|l|_h = \prod_i |x_i|^{-\nu_i/N}, \quad \text{where } x_i \text{ is the local defining equation of } D_i.$$

Then the induced singular Hermitian metric on $\mathcal{L}^{(j)^{-1}}(D^{(j)}) = \mathcal{L}^{-j}(\lfloor \frac{jD}{N} \rfloor + D^{(j)})$ locally is

$$\left| l^{-j} \prod_i x_i^{-\lfloor j\mu_i/N \rfloor} \prod_{j\nu_i/N \notin \mathbb{Z}} x_i^{-1} \right|_h = \prod_i |x_i|^{j\nu_i/N - \lfloor j\nu_i/N \rfloor} \prod_{j\nu_i/N\notin\mathbb{Z}} |x_i|^{-1} = \prod_i |x_i|^{-\{-j\nu_i/N\}}.$$

For any smooth top form Υ with values in $\mathcal{L}^{(j)^{-1}}(D^{(j)}) \otimes_{\mathbb{C}} \otimes \overline{\mathcal{L}^{(j)^{-1}}(D^{(j)})}$ we can associate an integrable top form $(\Upsilon)_h = f\overline{g}|s|_h^2 \operatorname{vol}(Z)$ by fixing a volume form $\operatorname{vol}(Z)$ on Z and writing locally $\Upsilon = fs \otimes \overline{gs} \operatorname{vol}(Z)$ for s a local fram of $\mathcal{L}^{(j)^{-1}}(D^{(j)})$. Therefore, we obtain a well-defined pairing,

$$\mathcal{A}_Z^k(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}} \wedge \overline{\mathcal{A}_Z^k(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}} \to \mathbb{C}, \quad (m', m'') \mapsto \frac{\varepsilon(\dim Z + k + 1)}{(2\pi\sqrt{-1})^{\dim Z}} \int_Z \left(\mathsf{X}^{\dim Z - k} m' \wedge \overline{m''}\right)_h. \tag{8.44}$$

Since $\eta : \tilde{Z} \to Z$ is generic finite, it follows from

$$\int_{\tilde{Z}} \eta^* \left(\mathsf{X}^{\dim Z-k} m' \wedge \overline{m''}\right) = N \int_Z \left(\mathsf{X}^{\dim Z-k} m' \wedge \overline{m''}\right)_h$$

that (8.44) induces the same polarization in the statement (2) of the above lemma except for the constant N.

Applying to our situation yields that $\mathscr{V}_{\alpha,J}(E^{\alpha,J})$ carries a canonical singular Hermitian metric $|-|_h$ with local weight functions $\prod_{j\in I\smallsetminus I_\alpha} |z_j|^{-\{\alpha e_j\}}$ restricted on Y^J, where z_i is the defining equation of Y_i. Provided the above two lemmas, the sum of hypercohomologies

$$\bigoplus_{k\in\mathbb{Z}} H^k \left(Y^J, \Omega_{Y^J}^{\bullet+\dim Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J}\right)$$

is a polarized Hodge-Lefschetz structure of central weight $\dim Y^J$ for any non-empty subset J of I_α. Similarly to Example 2.9 this is also determined by the filtered $\mathscr{D}_{Y^J}$-module $(\mathcal{V}_{\alpha,J}, F_\bullet\mathcal{V}_{\alpha,J})$ with the sesquilinear pairing $S_{\alpha,J} : \mathcal{V}_{\alpha,J} \otimes_{\mathbb{C}} \overline{\mathcal{V}_{\alpha,J}} \to \mathfrak{C}_{Y^J}$ is given by

$$([s_1 \otimes P_1], [s_2 \otimes P_2]) \mapsto \frac{\varepsilon(\dim Y^J + 1)}{(2\pi\sqrt{-1})^{\dim Y^J}} \int_{Y^J} (P_1 \overline{P_2} -)\, (s_1 \wedge \overline{s_2})_h \tag{8.45}$$

for local sections of $\mathcal{V}_{\alpha,J}$ (see (8.40)) represented by $s_i \otimes P_i$ such that s_i local sections of

$$\omega_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J} = \omega_{Y^J} \otimes \mathscr{V}_{\alpha,J}(E^{\alpha,J})$$

and P_i is a differential operator $i = 1,2$. Here, $(s_1 \wedge \overline{s_2})_h$ is the top form induced by the singular Hermitian metric on $\mathscr{V}_{\alpha,J}(E^{\alpha,J})$. Summarizing the results we proved in this subsection:

Corollary 8.4. *With notations as above, the direct sum of all hypercohomologies of the de Rham complex of $(\mathcal{V}_{\alpha,J}, F_\bullet\mathcal{V}_{\alpha,J})$ underlies a polarized Hodge-Lefschetz structure of central weight $\dim Y^J$ with the Hodge filtration induced by $F_\bullet\mathcal{V}_{\alpha,J}$ and with the polarization, on degree k, given by the following induced pairing scaled by $\varepsilon(k)$,*

$$H^k(Y^J, \mathrm{DR}_{Y^J}\mathcal{V}_{\alpha,J}) \otimes H^{-k}(Y^J, \mathrm{DR}_{Y^J}\mathcal{V}_{\alpha,J}) \longrightarrow H^0(Y^J, \mathrm{DR}_{Y^J,\overline{Y^J}}\mathcal{V}_{\alpha,J} \otimes_{\mathbb{C}} \overline{\mathcal{V}_{\alpha,J}}) \xrightarrow{S_{\alpha,J}} H^0(Y^J, \mathrm{DR}_{Y^J,\overline{Y^J}}\mathfrak{C}_{Y^J}) \simeq \mathbb{C}.$$

Remark 8.5. We cannot make the Hodge structure in the above corollary over $\mathbb{Q}$ because there is no eigenvalue decomposition of $\mathbb{Q}$-structure.

8.2. **Sesquilinear pairing.** As in the reduced case, we need a sesquilinear pairing to construct the limiting mixed Hodge structure. In fact, the construction for the reduced case still works with a little modification. Note that for any test function η over a local chart U and two local sections $\zeta_1 \otimes P_1, \zeta_2 \otimes P_2$ of $H^0\left(U, \Omega^n_{X/\Delta}(\log Y)(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X\right)$, the function

$$t \mapsto \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1\overline{P_2}(\eta)\zeta_1 \wedge \overline{\zeta_2}.$$

may have order approximately at most $|t|^{2\alpha}\left(-\log|t^2|\right)^k$ near $t = 0$ where $k+1$ is the number of components of Y_{I_α} that intersect in U. This suggests that we can define the pairing S_α on $\mathcal{M}_\alpha$ by

$$\langle S_\alpha([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle =_{\text{def}} \operatorname{Res}_{s=-\alpha} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} P_1\overline{P_2}(\eta) \frac{dt}{t} \wedge \zeta_1 \wedge \overline{\frac{dt}{t} \wedge \zeta_2}$$

$$= \operatorname{Res}_{s=-\alpha} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \overline{\frac{dt}{t}} \left(\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1\overline{P_2}(\eta)\zeta_1 \wedge \overline{\zeta_2} \right).$$

Again, we have not check that S_α is well-defined but let us do some local calculations to see what is going on.

Example 8.6. Suppose $Y = 2Y_0$ for Y_0 is a smooth manifold and t is equal to z_0^2 on X. Then R satisfies the equation $R(R - \frac{1}{2}) = 0$. We deduce that $\mathcal{M}$ has two eigenspaces $\mathcal{M}_0$ and $\mathcal{M}_{\frac{1}{2}}$ by (4.19). Then for any local sections $\zeta_i \otimes P_i = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes P_i$ of $\Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$, $i = 1, 2$ representing classes of $\mathcal{M}_0$, the calculation of the pairing $S_0([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2])$ is exactly as in the reduced case and as it turned out

$$S_0([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]) = i_{Y_0+}S_{Y_0}([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]).$$

By Theorem 7.3 $\mathcal{M}_{\frac{1}{2}}$ is locally generated by the class represented by $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes z_0$. Now for any local sections $\zeta \otimes z_0 P_i = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes z_0 P_i$ representing classes of $\mathcal{M}_{\frac{1}{2}}$, we have

$$\langle S_{\frac{1}{2}}([\zeta \otimes z_0 P_1], [\zeta \otimes z_0 P_2]), \eta\rangle = \operatorname{Res}_{s=-\frac{1}{2}} \int_X |z_0|^{4s} P_1\overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right)$$

$$= \int_X \frac{1}{2} \log|z_0|^2 \partial_0 \overline{\partial_0} P_1\overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right)$$

by Poincaré-Lelong equation [GH14, Page 388]

$$= \int_{Y_0} \frac{1}{2} P_1\overline{P_2}(\eta) \bigwedge_{i=1}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right)$$

$$= \frac{1}{2}\langle i_{Y_0+}S_{Y_0}([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle$$

$$= \frac{1}{2}\langle i_{Y_0+}S_{\frac{1}{2},\{0\}}([\zeta_1 \otimes z_0 P_1], [\zeta_2 \otimes z_0 P_2]), \eta\rangle,$$

Recall $S_{\frac{1}{2},\{0\}}$ defined in (8.45): since we have the isomorphism $\mathscr{O}_{Y_0}(2Y_0) = \mathscr{O}_{Y_0}(Y) \simeq \mathscr{O}_{Y_0}$ there exists a canonical singular Hermitian metric (this case is smooth) $|-|_h$ on $\mathscr{O}_{Y_0}(-Y_0)$ by setting the local frame z_0 has norm 1 so that

$$i_{Y_0+}S_{\frac{1}{2},\{0\}}([\zeta_1 \otimes z_0 P_1], [\zeta_2 \otimes z_0 P_2]), \eta\rangle = \int_X |z_0|_h^2 P_1\overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right) = i_{Y_0+}S_{Y_0}([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle.$$

The above equality can also be explained as follows: the cyclic covering constructed by taking out of the second root of the constant section of $\mathscr{O}_{Y_0}(2Y_0) \simeq \mathscr{O}_{Y_0}$ has two connected components and each component is isomorphic to Y_0.

Let η be a test function over an open subset U. For any two sections $m_1, m_2 \in H^0(U, \Omega^n_{X/\Delta}(\log Y)(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X)$, the $(2n+2)$-form $\frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}$ is smooth of outside Y and has pole along Y. Locally, the $(2n+2)$-form just is $|z_I|^{2\lceil\alpha\mathbf{e}\rceil}P_1\overline{P_2}(\eta)\frac{dt}{t}\wedge\zeta\wedge\overline{\frac{dt}{t}\wedge\zeta}$, where $m_j=\zeta\otimes z_I^{\lceil\alpha\mathbf{e}\rceil}P_j$ for $\zeta=\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge\cdots\wedge dz_n$ and $j=1,2$. Let $F(s)=F(s,m_1,m_2,\eta)$ be the meromorphic extension of

$$\frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}}\int_X |t|^{2s}\frac{dt}{t}\wedge m_1\wedge\overline{\frac{dt}{t}\wedge m_2}(\eta)$$

via integration by parts. The function $F(s)$ is well defined when $\operatorname{Re}s>-\alpha$ and has a pole at $s=-\alpha$. We only care about the polar part of $F(s)$ at $s=-\alpha$.

Theorem 8.7. *The polar part of $F(s)$ at $s=-\alpha$ is only depends on the classes of m_1 and m_2 in $\mathcal{M}_\alpha$.*

Proof. Let $\{\rho_\lambda\}$ be a partition of unity of the open covering $\{U_\lambda\}$ by local charts. Then

$$F(s)=\sum_\lambda\frac{\varepsilon(n+2)}{\left(2\pi\sqrt{-1}\right)^{n+1}}\int_{U_\lambda}|t|^{2s}\frac{dt}{t}\wedge m_1\wedge\overline{\frac{dt}{t}\wedge m_2}(\rho_\lambda\eta).$$

Since $\rho_\lambda\eta$ is a test function over local chart U_λ, we can assume U itself is a local chart. We assume $k+1$ components of Y intersect in U.

Lemma 8.8. *Under the assumption that $m_i=\zeta_\alpha\otimes P_i$ for $\zeta_\alpha=z_I^{\lceil\alpha\mathbf{e}\rceil}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n$ and for $i=1,2$, the followings are valid.*

(1) *the order of the pole of $F(s)$ at $s=-\alpha$ is at most $k+1$;*
(2) *if $P_i=t_\alpha P_i'$ for one of $i=1,2$, then $F(s)$ is holomorphic at $s=-\alpha$;*
(3) *for $0\le j\le k$ we have,*

$$F\left(s,\zeta_\alpha\otimes P_1,\zeta_\alpha\otimes\frac{1}{e_j}z_j\partial_jP_2,\eta\right)=F\left(s,\zeta_\alpha\otimes\frac{1}{e_j}z_j\partial_jP_1,\zeta_\alpha\otimes P_2,\eta\right)=\quad-\left(s+\frac{\lceil\alpha e_j\rceil}{e_j}\right)F(s,\zeta_1\otimes P_1,\zeta_2\otimes P_2,\eta).$$

Proof of the lemma. We work out Laurent series of $F(s)$ at $s=-\alpha$:

$$\begin{aligned}F(s)&=\int_X|z_I|^{2s\mathbf{e}+2\lceil\alpha\mathbf{e}\rceil-2\cdot\mathbf{1}}P_1\overline{P_2}(\eta)\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\\&=\int_X|z_I|^{2(s+\alpha)\mathbf{e}-2\cdot\mathbf{1}}|z_I|^{2\{-\alpha\mathbf{e}\}}P_1\overline{P_2}(\eta)\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\\&=\int_X(s+\alpha)^{-2(k+1)}|z_I|^{2(s+\alpha)\mathbf{e}}\eta'\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\quad\text{where }\eta'=\partial_I\overline{\partial_I}\left(|z_I|^{2\{-\alpha\mathbf{e}\}}P_1\overline{P_2}\eta\right)\\&=\sum_{\ell=0}^\infty\frac{1}{\ell!}(s+\alpha)^{\ell-2(k+1)}\int_X(\log|z_I|^{2\mathbf{e}})^\ell\eta'\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}).\end{aligned}$$

When $\ell<k+1$, then the form

$$(\log|z_I|^{2\mathbf{e}})^\ell\eta'\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}).$$

is actually exact because one of the a_i must be zero in the expansion of $(\log|z_I|^{2\mathbf{e}})^\ell$ into the linear combination of $\prod_{i=0}^k\left(\log|z_i|^{2e_i}\right)^{a_i}$ such that $\sum_{i=0}^k a_i=\ell$. Therefore, the order of the pole at $s=-\alpha$ is at most $k+1$.

When $P_1=t_\alpha P_1'$, the form

$$|z_I|^{2(s+\alpha)\mathbf{e}-2\cdot\mathbf{1}}|z_I|^{2\{-\alpha\mathbf{e}\}}t_\alpha P_1'\overline{P_2}(\eta)\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

is integrable when $s = -\alpha$ where $\{-\alpha\mathbf{e}\}$ is the multi-index such that $\{-\alpha\mathbf{e}\}_i = \{-\alpha e_i\}$. Therefore, $F(s)$ is holomorphic at $s = -\alpha$. It is the same when $P_2 = t_\alpha P_2'$.

Lastly, by linearity we can assume that $P_1 = P_2 = 1$.

$$
\begin{aligned}
F(s, \zeta_\alpha \otimes 1, \zeta_\alpha \otimes \frac{1}{e_j} z_j \partial_j, \eta) &= \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} \left(\frac{1}{e_j} \overline{z_j \partial_j \eta} \right) \frac{dt}{t} \wedge \zeta_\alpha \wedge \overline{\frac{dt}{t} \wedge \zeta_\alpha} \\
&= \int_X \prod_{i \in I \smallsetminus \{j\}} |z_i|^{2se_i + 2\lceil \alpha e_i \rceil - 2} z_j^{se_j + \lceil \alpha e_j \rceil - 1} \frac{1}{e_j} \overline{z_j^{se_j + \lceil \alpha e_j \rceil}} \partial_0 \eta \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right) \\
&= \int_X -\left(s + \frac{\lceil \alpha e_j \rceil}{e_j} \right) \prod_{i \in I} |z_i|^{2se_i + 2\lceil \alpha e_i \rceil - 2} \eta \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right) \\
&= -\left(s + \frac{\lceil \alpha e_j \rceil}{e_j} \right) \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} \eta \frac{dt}{t} \wedge \zeta_\alpha \wedge \overline{\frac{dt}{t} \wedge \zeta_\alpha}. \\
&= -\left(s + \frac{\lceil \alpha e_j \rceil}{e_j} \right) F(s, \zeta_\alpha \otimes 1, \zeta_\alpha \otimes 1, \eta).
\end{aligned}
\tag{8.46}
$$

The other equality in (3) holds similarly. We complete the proof of the lemma. □

Returning to the proof of theorem. Since $\mathcal{M}_\alpha$ is locally represented by

$$\zeta_\alpha \otimes \mathscr{D}_X / (t_\alpha, D_1 + \alpha_1, D_2 + \alpha_2, \ldots, D_n + \alpha_n) \mathscr{D}_X$$

(see the proof of Theorem 7.2), and (2) and (3) in the lemma say that when one of m_1 and m_2 is in

$$\zeta_\alpha \otimes (t_\alpha, D_1 + \alpha_1, D_2 + \alpha_2, \ldots, D_n + \alpha_n) \mathscr{D}_X$$

then $F(s)$ is holomorphic since α_i equals $\lceil \alpha e_i \rceil / e_i - \lceil \alpha e_0 \rceil / e_0$ for $1 \le i \le k$ and equals zero otherwise. □

For two sections $\gamma_1, \gamma_2 \in H^0(U, \mathcal{M})$ and any test function η over U, we define the pairing $S_\alpha : \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}_\alpha} \to \mathfrak{C}_X$ by

$$\langle S_\alpha(\gamma_1, \gamma_2), \eta \rangle = \mathrm{Res}_{s=-\alpha} \sum_\lambda F(s, \tilde{\gamma}_1, \tilde{\gamma}_2, \rho_\lambda \eta),$$

where $\{\rho_\lambda\}$ is a partition of unity with respect to an open covering by local charts $\{U_\lambda\}$ such that γ_i has a local lifting of $\tilde{\gamma}_i$ over U_λ for $i = 1, 2$. It is obvious that S_α is $\mathscr{D}_{X, \overline{X}}$-linear. Thus, it is a sesquilinear pairing. As a corollary of Lemma 8.8, we have

Corollary 8.9. *We have* $S_\alpha \circ (\mathrm{id} \otimes_{\mathbb{C}} R_\alpha) = S_\alpha \circ (R_\alpha \otimes_{\mathbb{C}} \mathrm{id})$.

Because of the corollary, the sesquilinear pairing S_α induces pairings on the associated graded quotient of the weight filtration

$$S_\alpha : \mathrm{gr}_k^W \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathrm{gr}_{-k}^W \mathcal{M}_\alpha} \to \mathfrak{C}_X,$$

as well as on the primitive part

$$P_{R_\alpha} S_r = S_\alpha \circ (\mathrm{id} \otimes_{\mathbb{C}} R_\alpha^r) : \mathcal{P}_{\alpha,r} \otimes_{\mathbb{C}} \overline{\mathcal{P}_{\alpha,r}} \to \mathfrak{C}_X.$$

Theorem 8.10. *The isomorphism* $\phi_{\alpha,r} : (\mathcal{P}_{\alpha,r}, F_\bullet \mathcal{P}_{\alpha,r}) \to \mathcal{V}_{\alpha,r}(-r)$ *in Theorem 7.13 respects the sesquilinear pairings up to a constant scalar. More concretely,*

$$P_{R_\alpha} S_r(m_1, m_2) = \bigoplus_{\substack{J \subset I_\alpha, \\ \# J = r+1}} \frac{(-1)^r}{(r+1)! C_J} \tau_+^J S_{\alpha,J}(\phi_{\alpha,r} m_1, \phi_{\alpha,r} m_2)$$

for any local sections $m_1, m_2 \in \mathcal{P}_{\alpha,r}$ and $C_J = \prod_{j\in J} e_j$, where the pairing $S_{\alpha,J} : \mathcal{V}_{\alpha,J} \otimes_{\mathbb{C}} \overline{\mathcal{V}_{\alpha,J}} \to \mathfrak{C}_{Y^J}$ is defined in (8.45).

Proof. Because of the linearity and the generators of $\mathcal{P}_{\alpha,r}$ are all monomials dividing t_α of degree $\mu - r$ Corollary 7.7, it suffices to prove the theorem in the case when m_i is represented by

$$\zeta_\alpha \otimes z_{K_i} = z_I^{\lceil \alpha \mathbf{e} \rceil} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n \otimes z_{K_i}$$

where $K_i \subset I_\alpha$ with $\# K_i = \mu - r$ and $i = 1, 2$. Let η be a test function over U. Then we have

$$\langle S_\alpha(m_1, R_\alpha^r m_2), \eta \rangle = \mathrm{Res}_{s=-\alpha}(-(s+\alpha))^r \int_X |z_I|^{2s\mathbf{e}+2\lceil \alpha \mathbf{e} \rceil - 2\cdot \mathbf{1}} z_{K_1} \overline{z_{K_2}} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i}).$$

If $m_1 \neq m_2$, then the above is zero. Indeed, for $v \in K_2 \setminus K_1$ by choosing $R_\alpha^r = 1 \otimes \prod_{i\in I\setminus K_1 \setminus \{v\}} \frac{1}{e_i} z_i \partial_i$,

$$\langle S(R_\alpha^r m_1, m_2), \eta \rangle = \mathrm{Res}_{s=-\alpha} \int_X |z_I|^{2s\mathbf{e}-2\cdot\mathbf{1}} |z_I|^{2\lceil \alpha \mathbf{e} \rceil} \frac{t_\alpha}{z_v} \overline{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

where $\tilde{\eta} = C_{I\setminus K_1 \setminus \{v\}}^{-1} \partial_{I\setminus K_1\setminus\{v\}} \overline{z_{K_2}} (\overline{z_v})^{-1} \eta$ is a smooth function with compact support. The function

$$\int_X |z_I|^{2s\mathbf{e}-2\cdot\mathbf{1}} |z_I|^{2\lceil \alpha \mathbf{e} \rceil} \frac{t_\alpha}{z_v} \overline{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

is holomorphic at $s = -\alpha$ because by setting $s = -\alpha$ the form

$$|z_I|^{-2\alpha\mathbf{e}-2\cdot\mathbf{1}} |z_I|^{2\lceil \alpha \mathbf{e} \rceil} \frac{t_\alpha}{z_v} \overline{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) = |z_{I\setminus I_\alpha}|^{-2\{\alpha \mathbf{e}\}} \frac{1}{\overline{t_\alpha}} \frac{\overline{z}_v}{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

is integrable.

Therefore, we reduce the proof to the case when $m_1 = m_2 = m$ represented by $\zeta_\alpha \otimes z_K$. We shall prove that

$$S_\alpha(m, R_\alpha^r m) = \frac{(-1)^r}{(r+1)! C_{\overline{K}}} \tau_+^{\overline{K}} S_{\alpha, \overline{K}}(\phi_{\alpha,r} m, \phi_{\alpha,r} m),$$

where $\overline{K}$ is the complement of K in I_α. Without loss of generality, we can assume that $K = \{r+1, r+2, ..., \mu\}$ and $\overline{K} = \{0, 1, ..., r\}$ so that $z_K = z_{r+1} z_{r+2} \cdots z_\mu$. We have

$$\langle S(m, R_\alpha^r m), \eta \rangle = \mathrm{Res}_{s=-\alpha}(-(s+\alpha))^r \int_X |z_K|^{2(s+\alpha)\mathbf{e}_K} |z_{I\setminus K}|^{2s\mathbf{e}_{I\setminus K} + 2\lceil \alpha \mathbf{e}_{I\setminus K} \rceil - 2} \eta \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \tag{8.47}$$

where, for any index subset $J \subset I$, the j-th component the multi-index $\mathbf{e}_J$ is e_j if $j \in J$ or zero otherwise, and the j-th component of $\lceil \alpha \mathbf{e}_J \rceil$ is $\lceil \alpha e_j \rceil$ if $j \in J$ or zero otherwise. Integration by parts for $\{dz_i, d\bar{z}_i\}_{i \in \overline{K}}$, the identity (8.47) equals to

$$\mathrm{Res}_{s=-\alpha}(-(s+\alpha))^r \int_X \frac{|z_{I_\alpha}|^{2(s+\alpha)\mathbf{e}_{I_\alpha}}}{C_{\overline{K}}^2 (s+\alpha)^{2r+2}} |z_{I\setminus I_\alpha}|^{2s\mathbf{e}_{I\setminus I_\alpha} + 2\lceil \alpha \mathbf{e}_{I\setminus I_\alpha} \rceil - 2} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta\right) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \tag{8.48}$$

$$= \mathrm{Res}_{s=-\alpha} \frac{(-1)^r}{C_{\overline{K}}^2 (s+\alpha)^{r+2}} \int_X |t|^{2(s+\alpha)} \prod_{j\in I\setminus I_\alpha} |z_j|^{-2\{\alpha e_j\}} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta\right) \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \tag{8.49}$$

where $\partial_{\overline{K}} \overline{\partial_{\overline{K}}} = \prod_{j\in\overline{K}} \partial_j \overline{\partial_j}$. Because of the expansion

$$|t|^{2(s+\alpha)} = \exp\left(\log |t|^2 \, (s+\alpha)\right) = \sum_{\ell=0}^{\infty} \frac{\left(\log |t|^2\right)^\ell (s+\alpha)^\ell}{\ell!},$$

we find that (8.49) is equal to

$$\frac{(-1)^r}{C_{\overline{K}}^2(r+1)!}\int_X \left(\log|t|^2\right)^{r+1}\prod_{j\in I\smallsetminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\left(\partial_{\overline{K}}\overline{\partial_{\overline{K}}}\eta\right)\bigwedge_{i=0}^n\left(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}\right) \tag{8.50}$$

The expansion of $\left(\log|t|^2\right)^{r+1}$ is a linear combination of

$$\prod_{i\in I}\left(\log|z_i|^2\right)^{a_i}$$

for all partitions $\sum_{i\in I}a_i = r+1$, but the differential form

$$\prod_{i\in I}\left(\log|z_i|^2\right)^{a_i}\prod_{j\in I\smallsetminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\left(\partial_{\overline{K}}\overline{\partial_{\overline{K}}}\eta\right)\bigwedge_{i=0}^n\left(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}\right)$$

is exact unless $a_i \neq 0$ for any $i\in\overline{K}$, which is equivalent to $a_i = 1$ for $i\in\overline{K}$ and $a_i = 0$ for $i\notin\overline{K}$. It follows that (8.50) is equal to

$$\frac{(-1)^r}{C_{\overline{K}}(r+1)!}\int_X\prod_{j\in\overline{K}}\log|z_j|^2\prod_{j\in I\smallsetminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\left(\partial_{\overline{K}}\overline{\partial_{\overline{K}}}\eta\right)\bigwedge_{i=0}^n\left(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}\right).$$

We deduce from Poincáre-Lelong equation [GH14, Page 388] that the above continues to equal to

$$\frac{(-1)^r}{(r+1)!C_{\overline{K}}}\int_{Y^{\overline{K}}}\prod_{j\in I\smallsetminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\eta\bigwedge_{i=r+1}^n\left(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}\right) \tag{8.51}$$

Since $\phi_{\alpha,\overline{K}}m = \pm\frac{dz_{I\smallsetminus K}}{z_{I\smallsetminus K}}\wedge dz_{k+1}\wedge\cdots\wedge dz_n\otimes s_{\alpha,\overline{K}}\in\omega_{Y^{\overline{K}}}(E^{\alpha,\overline{K}})\otimes\mathscr{V}_{\alpha,\overline{K}}$, it follows that

$$(\phi_{\alpha,\overline{K}}m\wedge\overline{\phi_{\alpha,\overline{K}}m})_h = \prod_{j\in I\smallsetminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\bigwedge_{i=r+1}^n\left(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}\right)$$

from which we conclude that (8.51) is equal to

$$\frac{(-1)^r}{(r+1)!C_{\overline{K}}}\int_{Y^{\overline{K}}}\eta(\phi_{\alpha,\overline{K}}m\wedge\overline{\phi_{\alpha,\overline{K}}m})_h = \frac{(-1)^r}{(r+1)!C_{\overline{K}}}\langle S_{\alpha,\overline{K}}(\phi_{\alpha,\overline{K}}m,\phi_{\alpha,\overline{K}}m),\eta\rangle.$$

See (8.45). The theorem is proved. □

8.3. **Construction of the limiting mixed Hodge structure.** We begin to construct a polarized bigraded Hodge-Lefschetz structure on $\mathrm{gr}^W H^\bullet(X,\mathrm{DR}_X\mathcal{M}_\alpha)$. Fix a Kähler class ω on X and let $L = \omega\wedge : \mathrm{DR}_X\mathcal{M}_\alpha \to \mathrm{DR}_X\mathcal{M}_\alpha[2]$ be the Lefschetz operator and $\mathsf{X}_1 = 2\pi\sqrt{-1}L$. Relabel the graded pieces of the first page of the weight spectral sequence by

$$V^\alpha_{\ell,k} = H^\ell(X,\mathrm{gr}^W_k\mathrm{DR}_X\mathcal{M}_\alpha) = {}^W E_1^{-k,\ell+k}.$$

Let $V^\alpha = \bigoplus_{\ell,k\in\mathbb{Z}}V^\alpha_{\ell,k}$ with the filtration $F_\bullet V^\alpha$ induced by $F_\bullet\mathcal{M}_\alpha$. Denote by $E_i(R_\alpha)$ the induced operator by R_α on ${}^W E_i$ and let $\mathsf{Y}_2 = E_1(R_\alpha)$. Denote by $S_{\ell,k}$ the induced pairing on $V^\alpha_{\ell,k}\otimes\overline{V^\alpha_{-\ell,-k}}$

$$H^\ell(X,\mathrm{gr}^W_k\mathrm{DR}_X\mathcal{M}_\alpha)\otimes\overline{H^{-\ell}(X,\mathrm{gr}^W_{-k}\mathrm{DR}_X\mathcal{M}_\alpha)}\to H^0(X,\mathrm{DR}_{X,\overline{X}}\mathrm{gr}^W_k\mathcal{M}_\alpha\otimes_{\mathbb{C}}\overline{\mathrm{gr}^W_{-k}\mathcal{M}_\alpha})\to H^0_c(X,\mathrm{DR}_{X,\overline{X}}\mathfrak{C}_X)\simeq\mathbb{C},$$

modifying by a sign factor $\varepsilon(\ell)$. Let d_1 be the differential of the first page of the spectral sequence. In terms of relabeling we have

$$d_1 : (V^\alpha_{\ell,k},F_\bullet V^\alpha_{\ell,k})\to(V^\alpha_{\ell+1,k-1},F_\bullet V^\alpha_{\ell+1,k-1}).$$

Exactly same to Theorem 6.6 and Corollary 6.7 in the reduced case, we conclude that

Theorem 8.11. *The tuple* $(V^\alpha, \mathsf{X}_1, \mathsf{Y}_2, F_\bullet V, \bigoplus S_{j,k}, d_1)$ *gives a differential polarized bigraded Hodge-Lefschetz structure of central weight* n.

Corollary 8.12. *We have the following*

1. *Hodge spectral sequence degenerates at* ${}_F E_1$*;*
2. *the weight spectral sequence degenerates at* ${}^W E_2$*;*
3. *the tuple* $(\bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}^W H^\ell(X, \mathrm{DR}_X \mathcal{M}_\alpha), \mathsf{X}_1, \mathsf{Y}_2, F_\bullet)$ *together with the pairing induced by* S_α *is a polarized bigraded Hodge-Lefschetz structure of central weight* n.

The last statement in the above corollary implies that the induced weight filtration on $H^\ell(X, \mathrm{DR}_X\mathcal{M}_\alpha)$ is the monodromy filtration associated to R_α on $H^\ell(X, \mathrm{DR}_X\mathcal{M}_\alpha)$. We established Theorem A.

9. Application

9.1. **Hard Lefschetz.** The following is a consequence of the bigraded Hodge-Lefschetz structure

Theorem 9.1. *The Lefschetz operator induces an isomorphism between* $\mathscr{O}_\Delta$*-modules*

$$\left(2\pi\sqrt{-1}L\right)^k : F_\ell R^{-k}\Omega^{\bullet+n}_{X/\Delta}(\log Y) \simeq F_{\ell-k}R^k\Omega^{\bullet+n}_{X/\Delta}(\log Y) \quad \textit{for any integer } \ell.$$

As a result, we have the following decomposition in the derived category of coherent $\mathscr{O}_\Delta$*-modules:*

$$Rf_*F_\ell\Omega^{\bullet+n}_{X/\Delta}(\log Y) \simeq \bigoplus_{k\in\mathbb{Z}} F_\ell R^k f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)[-k] \quad \textit{for any integer } \ell.$$

Proof. The first statement follows from the Hard Lefschetz on each fiber

$$\left(2\pi\sqrt{-1}L\right)^k : F_\ell R^{-k}\Omega^{\bullet+n}_{X/\Delta}(\log Y)\otimes\mathbb{C}(p) \simeq F_{\ell-k}R^k\Omega^{\bullet+n}_{X/\Delta}(\log Y)\otimes\mathbb{C}(p),$$

for every $p\in\Delta$. The second statement follows from the first one plus the main theorem in [Del68]. □

9.2. **Invariant cycle theorem.** Now we shall give the proof of Theorem B, which is equivalently to the following statement:

Theorem 9.2. *We have the following exact sequence of mixed Hodge structures*

$$H^\ell(Y,\mathbb{C}) \longrightarrow H^\ell(X, \mathrm{DR}_X\mathcal{M}) \xrightarrow{R} H^\ell(X, \mathrm{DR}_X\mathcal{M})(-1).$$

Of course one can try to show that $\ker R$ is the filtered $\mathscr{D}_X$-module such that the hypercohomologies of its de Rham complex computes the cohomologies of Y. But we would like to keep the proof elementary so we will just show that the first page of the weight spectral sequence computing the hypercohomology of $\mathrm{DR}_X \ker R$ is the same to the one computing the cohomology of Y up to a constant scalar; this will prove the theorem because both weight spectral sequences degenerate at the second page. See [GS75, (4.2)] or [Ste76, (3.5)] for the weight filtration of $H^\ell(Y,\mathbb{C})$

Proof. Note that $\ker R$ is contained in $\mathcal{M}_0$. Therefore, $W_{-j}\ker R = R^j\ker R^{j+1}$ for $j\geq 0$ and vanishes for $j<0$ where $W = W(R)$ on $\mathcal{M}_0$. It follows that $\mathrm{gr}^W_{-j}\ker R$ is isomorphic to $\omega_{\tilde{Y}(j+1)}$ for $j\geq 0$ by Theorem 7.13. Because $\mathrm{gr}^W_{-j}\ker R$ is a summand of $\mathrm{gr}^W_{-j}\mathcal{M}_0$ for $j\geq 0$ by the Lefschetz decomposition on $\mathrm{gr}^W\mathcal{M}_0$, we have the following short exact sequence of Hodge structures on the first page of the weight spectral sequences:

$$0 \longrightarrow H^{\ell+\bullet}(X, \mathrm{gr}^W_{-j-\bullet}\mathrm{DR}_X\ker R) \longrightarrow H^{\ell+\bullet}(X, \mathrm{gr}^W_{-j-\bullet}\mathrm{DR}_X\mathcal{M}_0) \xrightarrow{R} H^{\ell+\bullet}(X, \mathrm{gr}^W_{-j-2-\bullet}\mathrm{DR}_X\mathcal{M}_0)(-1) \longrightarrow 0.$$

The associated long exact sequence gives the relation between the second page of the spectral sequences:

$$\cdots \longrightarrow \mathrm{gr}^W_{-j} H^\ell(X, \mathrm{DR}_X \ker R) \longrightarrow \mathrm{gr}^W_{-j} H^\ell(X, \mathrm{DR}_X \mathcal{M}_0) \longrightarrow \mathrm{gr}^W_{-j-2} H^\ell(X, \mathrm{DR}_X \mathcal{M}_0)(-1) \longrightarrow \cdots.$$

Now it remains to prove that $H^\ell(X, \mathrm{DR}_X \ker R)$ and $H^\ell(Y, \mathbb{C})$ are isomorphic as mixed Hodge structures. It suffices to check that they coincide at the first page of weight spectral sequence since they degenerate at the second page. We have the following commutative diagram where the leftmost column is the E_1-page spectral sequence of $\ker R$ and all the horizontal arrows are isomorphisms of mixed Hodge structures.

$$\begin{array}{ccccc} H^\ell(X, \mathrm{gr}^W_{-j} \mathrm{DR}_X \ker R) & \xrightarrow{\phi_{0,r} \circ (R^j)^{-1}} & H^\ell(X, \mathrm{DR}_X \tau^{j+1}_+ \omega_{\tilde{Y}^{(j+1)}}) & \xleftarrow{\simeq} & H^\ell(\tilde{Y}^{(j+1)}, \Omega^{n-j+\bullet}_{\tilde{Y}^{(j+1)}}) \\ \Big\downarrow d_1 & & \Big\downarrow & & \Big\downarrow \\ H^{\ell+1}(X, \mathrm{gr}^W_{-(j+1)} \mathrm{DR}_X \ker R) & \xrightarrow{\simeq} & H^{\ell+1}(X, \mathrm{DR}_X \tau^{j+2}_+ \omega_{\tilde{Y}^{(j+2)}}) & \xleftarrow{\simeq} & H^{\ell+1}(\tilde{Y}^{(j+2)}, \Omega^{n-j+1+\bullet}_{\tilde{Y}^{(j+2)}}) \end{array} \tag{9.52}$$

We shall identify the the rightmost vertical arrow with the differential of the first page of the weight spectral sequence of $H^\ell(Y, \mathbb{C})$ via diagram chasing.

$$\begin{array}{ccccc} \mathrm{gr}^W_{-j} \ker R \otimes \bigwedge^p \mathscr{T}_X & \xrightarrow{\simeq} & \tau^K_+ \omega_{Y^K} \otimes \bigwedge^p \mathscr{T}_X & \longleftarrow & \Omega^{n-j-p}_{Y^K} \\ \Big\downarrow d & & \Big\downarrow & & \Big\downarrow \\ \mathrm{gr}^W_{-(j+1)} \ker R \otimes \bigwedge^{p-1} \mathscr{T}_X & \xrightarrow{\simeq} & \bigoplus_{j_i \in J} \tau^{K \cap \{j_i\}}_+ \omega_{Y^{K \cap \{j_i\}}} \otimes \bigwedge^{p-1} \mathscr{T}_X & \longleftarrow & \bigoplus_{j_i \in J} \Omega^{n-j-p}_{Y^{K \cap \{j_i\}}} \end{array}$$

$$\begin{array}{ccccc} \left[\pm R^j \zeta_0 \otimes z_I z_K^{-1} \otimes \partial_J\right] & \longrightarrow & \pm dz_{\bar{K}} \otimes \partial_J & \longleftarrow & dz_{\bar{K} \smallsetminus J} \\ \Big\downarrow d & & \Big\downarrow & & \Big\downarrow \\ \left[\pm R^{j+1} \zeta_0 \otimes \sum_{j_i \in J} e_{j_i} z_I z_K {z_{j_i}}^{-1} \otimes \partial_{J \smallsetminus \{j_i\}}\right] & \longrightarrow & \bigoplus_{j_i \in J} \pm dz_{\bar{K}} \otimes \partial_{J \smallsetminus \{j_i\}} & \longleftarrow & \pm \sum_{j_i \in J} e_{j_i} dz_{\bar{K} \smallsetminus J} \end{array}$$

Starting from the upper-right corner, let $dz_{\bar{K} \smallsetminus J} = \bigwedge_{i \in \bar{K} \smallsetminus J} dz_i$ be a local section of $\Omega^{n-j-p}_{Y^K}$ where K is an ordered index set of cardinality $j+1$, $\bar{K}$ is the complement of K in I and $J \subset \bar{K}$ of cardinality p. Then $\pm dz_{\bar{K}} \otimes \partial_J$ is the image in $\tau^K_+ \omega_{Y^K} \otimes \bigwedge^p \mathscr{T}_X$ via the inclusion

$$\Omega^{n-j-p}_{Y^K} = \omega_{Y^K} \otimes \bigwedge^p \mathscr{T}_{Y^K} \to \tau^K_+ \omega_{Y^K} \otimes \bigwedge^p \mathscr{T}_X,$$

where $\partial_J = \bigwedge_{j \in J} \partial_j$. Its preimage under the isomorphism

$$\phi_{0,K} \circ \left(R^j\right)^{-1} : \mathrm{gr}^W_j \ker R \otimes \bigwedge^p \mathscr{T}_X = R^j \ker R^{j+1} \otimes \bigwedge^p \mathscr{T}_X \to \mathcal{P}_{0,-j} \otimes \bigwedge^p \mathscr{T}_X \to \tau^K_+ \omega_{Y^K} \otimes \bigwedge^p \mathscr{T}_X$$

is the class represented by $\pm R^j \zeta_0 \otimes z_I z_K^{-1} \otimes \partial_J$, where $\zeta_0 = \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n$ and $\mathcal{P}_{0,-j}$ is the $(-j)$th-primitive part of $\mathrm{gr}^W \mathcal{M}_0$. It maps to the class of $\pm R^{j+1} \zeta_0 \otimes \sum_{j_i \in J} e_{j_i} z_I (z_K z_{j_i})^{-1} \otimes \partial_{J \smallsetminus \{j_i\}}$ by the differential of $\mathrm{DR}_X \ker R$. By reverse the above procedure, $\pm R^{j+1} \zeta_0 \sum_{j_i \in J} e_{j_i} z_I (z_K z_{j_i})^{-1} \otimes \partial_{J \smallsetminus \{j_i\}}$ corresponds to $\pm \sum_{j_i \in J} e_{j_i} dz_{\bar{K} \smallsetminus J}$ restricting on $\bigoplus_{j_i \in J} \Omega^{n-j-i-p}_{Y^{K \cap \{j_i\}}}$. Therefore, the morphism d_1 in the diagram (9.52), up to a scalar factor, can be identified with the pullback

$$H^\ell\left(\tilde{Y}^{(j+1)}, \Omega^{n-j+\bullet}_{\tilde{Y}^{(j+1)}}\right) \to H^{\ell+1}\left(\tilde{Y}^{(j+2)}, \Omega^{n-j-1+\bullet}_{\tilde{Y}^{(j+2)}}\right),$$

which is the differential of the ${}^W E_1$-page of $H^\ell(Y, \mathbb{C})$. This completes the proof. ◻

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