THE MINIMAL EXPONENT AND k-RATIONALITY FOR LOCALLY COMPLETE INTERSECTIONS

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ABSTRACT. We show that if Z is a locally complete intersection subvariety of a smooth complex variety X, of pure codimension r, then Z has k-rational singularities if and only if $\widetilde{\alpha}(Z) > k + r$, where $\widetilde{\alpha}(Z)$ is the minimal exponent of Z. We also characterize this condition in terms of the Hodge filtration on the intersection cohomology Hodge module of Z. Furthermore, we show that if Z has k-rational singularities, then the Hodge filtration on the local cohomology sheaf $\mathcal{H}^r_Z(\mathcal{O}_X)$ is generated at level $\dim(X) - \lceil \widetilde{\alpha}(Z) \rceil - 1$ and, assuming that $k \geq 1$ and Z is singular, of dimension d, that $\mathcal{H}^k(\underline{\Omega}_Z^{d-k}) \neq 0$. All these results have been known for hypersurfaces in smooth varieties.

1. INTRODUCTION

It is well-known that rational and Du Bois singularities play an important role in the hierarchy of singularities of higher-dimensional algebraic varieties. Recently, definitions of "higher order" versions of these classes of singularities have been proposed, as follows. Suppose that Z is a complex algebraic variety. If $\underline{\Omega}_Z^p$ is the p-th graded piece of the Du Bois complex of Z (suitably shifted), then there is a canonical morphism

$$\Omega_Z^p \to \underline{\Omega}_Z^p$$

that is an isomorphism over the smooth locus of Z. Following [JKSY21], we say that Z has k-Du Bois singularities if this morphism is an isomorphism for $0 \leq p \leq k$. For $k = 0$, we recover the definition of Du Bois singularities.

On the other hand, if $\mu\colon \widetilde{Z} \to Z$ is a resolution of singularities that is an isomorphism over $Z \smallsetminus Z_{\rm sing}$ and such that $D = \mu^{-1}(Z_{\rm sing})$ is a simple normal crossing divisor on $\widetilde{Z}$, then following [FL22a] we say that Z has k-rational singularities if the canonical morphism

$$\Omega_Z^p \to \mathbf{R}\mu_*\Omega_{\widetilde{Z}}^p(\log D)$$

is an isomorphism for $0 \leq p \leq k$. Again, for $k = 0$ this is the classical notion of rational singularities. Our main goal in this note is to characterize numerically, in the case when Z is locally a complete intersection, the condition for having k-rational singularities. A similar characterization for k-Du Bois locally complete intersections has been obtained in [MP22a], extending work on hypersurfaces in [MOPW21] and [JKSY21].

Suppose that X is a smooth, irreducible, n-dimensional complex algebraic variety and Z is a locally complete intersection closed subscheme of X, of pure codimension r in X. In this setting the *minimal exponent* $\widetilde{\alpha}(Z)$ was introduced and studied in [CDMO22]. In the case $r = 1$, this is the invariant introduced by Saito in [Sai94] as the negative of the largest root of the reduced Bernstein-Sato polynomial of Z. In general, $\widetilde{\alpha}(Z)$ can be described

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in terms of the Kashiwara-Malgrange V-filtration associated to Z and it is also related to the Hodge filtration on the local cohomology sheaf $\mathcal{H}^r_Z(\mathcal{O}_X)$. The minimal exponent can be considered as a refinement of the log canonical threshold of (X,Z): we always have $\mathrm{lct}(X,Z)=\min\{\widetilde{\alpha}(Z),r\}$. Moreover, it is shown in [CDMO22] that $\widetilde{\alpha}(Z)>r$ if and only if Z has rational singularities, extending a result due to Saito [Sai93] in the case of hypersurfaces.

The following is our main result:

Theorem 1.1. *If Z is a locally complete intersection subvariety of the smooth, irreducible variety X, of pure codimension r, then Z has k-rational singularities if and only if $\widetilde{\alpha}(Z)>k+r$.*

In the case of hypersurfaces, this result was proved independently in [FL22b] and [MP22b]. The proof we give follows the idea in [FL22b], making also essential use of results from [CD21] on the Kashiwara-Malgrange V-filtration in the case of higher codimension subvarieties. A key ingredient in the proof is Saito's theory of mixed Hodge modules [Sai90].

The characterization of k-Du Bois singularities in [MP22a] for locally complete intersections can also be formulated in terms of the minimal exponent: it says that, with the notation in Theorem 1.1, Z has k-Du Bois singularities if and only if $\widetilde{\alpha}(Z)\geq k+r$. In particular, we obtain the following

Corollary 1.2. *If Z is a complex algebraic variety which is locally a complete intersection and if Z has k-Du Bois singularities, for some $k\geq 1$, then Z has $(k-1)$-rational singularities.*

Another consequence of the numerical characterizations of k-rational and k-Du Bois locally complete intersection singularities is that k-rational implies k-Du Bois. However, this result has already been known (it was proved independently in [FL22b] and [MP22a]) and we use it in our proof of Theorem 1.1.

As a consequence of the result in Theorem 1.1 and of general properties of the minimal exponent, we obtain an upper bound for the dimension of the singular locus. We note that if in the following corollary we replace "k-rational" by "k-Du Bois", then it follows from the results in [MP22a] that $\mathrm{codim}_Z(Z_{\mathrm{sing}})\geq 2k+1$.

Corollary 1.3. *If Z is a complex algebraic variety which is locally a complete intersection and if Z has k-rational singularities, then*

$$\mathrm{codim}_Z(Z_{\mathrm{sing}})\geq 2k+2.$$

Let us recall the condition for k-Du Bois singularities in terms of the Hodge filtration on local cohomology. For every subvariety Z of a smooth complex algebraic variety X and every i, the local cohomology sheaf $\mathcal{H}^i_Z(\mathcal{O}_X)$ underlies a mixed Hodge module. As such, it carries a Hodge filtration $F_\bullet\mathcal{H}^i_Z(\mathcal{O}_X)$, an increasing filtration by coherent $\mathcal{O}_X$-submodules. If Z is a locally complete intersection of pure codimension r, then the only nonzero local cohomology sheaf is $\mathcal{H}^r_Z(\mathcal{O}_X)$. There is another filtration $E_\bullet\mathcal{H}^r_Z(\mathcal{O}_X)$ on $\mathcal{H}^r_Z(\mathcal{O}_X)$, also by coherent $\mathcal{O}_X$-modules, given by

$$E_p\mathcal{H}^r_Z(\mathcal{O}_X)=\big\{u\in\mathcal{H}^r_Z(\mathcal{O}_X)\mid I_Z^{p+1}u=0\big\}\quad\text{for}\quad p\geq 0,$$

where I_Z is the ideal defining Z. It is shown in [MP22a] that $F_p\mathcal{H}^r_Z(\mathcal{O}_X)\subseteq E_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for all $p\geq 0$ and equality for $p=k$ implies equality also for $p<k$. One defines the *cohomological level* of the Hodge filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$ by

$$p(Z)=\sup\big\{k\geq 0\mid F_k\mathcal{H}^r_Z(\mathcal{O}_X)=E_k\mathcal{H}^r(\mathcal{O}_X)\big\},$$

with the convention that $p(Z) = -1$ if there are no such k. It is then shown in [MP22a] that Z has k-Du Bois singularities if and only if $p(Z) \geq k$. The condition in terms of the minimal exponent follows from this and the equality $p(Z) = \max\big\{\lfloor\widetilde{\alpha}(Z)\rfloor - r, -1\big\}$, proved in [CDMO22].

We characterize k-rationality in a similar fashion. Recall that if X is a smooth irreducible n-dimensional variety and Z is a closed subvariety of X of pure codimension r, then $\mathcal{H}^r_Z(\mathcal{O}_X)$ also carries a weight filtration and the lowest weight piece is $W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)$, which underlies a pure Hodge module of weight $n+r$ (this $\mathcal{D}_X$-module is the *intersection cohomology* $\mathcal{D}_X$-module of Brylinski and Kashiwara [BK81]). We prove the following result, which in the case of hypersurfaces was proved in [Ola22].

Theorem 1.4. *If Z is a locally complete intersection subvariety of the smooth, irreducible, n-dimensional variety X, of pure codimension r, then for every nonnegtive integer k, we have $\widetilde{\alpha}(Z) > k + r$ if and only if $F_kW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = E_k\mathcal{H}^r_Z(\mathcal{O}_X)$.*

We also show that for singular locally complete intersections that have k-rational singularities, with $k \geq 1$, some higher cohomology groups of the graded pieces of the Du Bois complex do not vanish. This extends the result from [MOPW21, Theorem 1.5] in the case of hypersurfaces.

Theorem 1.5. *Let Z be a locally complete intersection subvariety of the smooth, irreducible, n-dimensional variety X. If Z has pure dimension d and k-rational singularities, for some $k \geq 1$, then*

$$\mathcal{H}^k\big(\underline{\Omega}_Z^{d-k}\big) \simeq \mathcal{E}xt^k_{\mathcal{O}_Z}(\Omega^k_Z, \omega_Z) \simeq \omega_Z \otimes_{\mathcal{O}_Z} \mathrm{Sym}^k_{\mathcal{O}_Z}\mathcal{Q},$$

where $\mathcal{Q}$ is the cokernel of the canonical map $\mathcal{T}_X|_Z \to \mathcal{N}_{Z/X}$. In particular, if Z is singular at x, then $\mathcal{H}^k\big(\underline{\Omega}_Z^{d-k}\big)_x \neq 0$.

As observed in [MOPW21], such a result imposes restrictions on varieties with quotient or toroidal singularities. Indeed, if Z is a variety with quotient or toroidal singularities, then $\mathcal{H}^i\big(\underline{\Omega}_Z^p\big) = 0$ for all p and all $i \geq 1$; for quotient singularities, this follows from [DB81, Section 5] and for toroidal singularities, it follows from [GNAPGP88, Chapter V.4]. On the other hand, it is well-known that such singularities are rational. By combining Theorems 1.1 and 1.5, we thus obtain

Corollary 1.6. *Let Z be a locally complete intersection subvariety, of pure codimension r, of the smooth, irreducible algebraic variety X. If Z is singular, with quotient or toroidal singularities, then $r < \widetilde{\alpha}(Z) \leq r + 1$.*

Our final result concerns the level of generation of the Hodge filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$. Recall that if $\mathcal{M}$ is a $\mathcal{D}_X$-module endowed with a good filtration, where $\mathcal{D}_X$ is the sheaf of differential operators on X, then we have $F_1\mathcal{D}_X \cdot F_p\mathcal{M} \subseteq F_{p+1}\mathcal{M}$, with equality for $p \gg 0$ (here $F_\bullet\mathcal{D}_X$ is the order filtration on $\mathcal{D}_X$). If equality holds for $p \geq p_0$, we say that the filtration on $\mathcal{M}$ is generated at level p_0. This definition applies, in particular, for the filtered $\mathcal{D}_X$-module underlying a mixed Hodge module on X.

Theorem 1.7. *If Z is a singular, pure codimension r, locally complete intersection subvariety of the smooth, irreducible, n-dimensional variety X, then the Hodge filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$ is generated at level $n - \lceil\widetilde{\alpha}(Z)\rceil - 1$.*

When $r = 1$, this is [MP20, Theorem A]. We also note that it follows from [MP22a, Theorem 4.2] that the filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$ is always generated at level $n - r$, hence the assertion in the above theorem is interesting when $\widetilde{\alpha}(Z) > r-1$. Furthermore, via the equivalence in *loc. cit.*, the assertion in Theorem 1.7 admits the following interpretation in terms of relative vanishing.

Theorem 1.8. *Let Z be a singular, pure codimension r, locally complete intersection subvariety of the smooth, irreducible, n-dimensional variety X. If $f\colon Y\to X$ is a proper morphism that is an isomorphism over $X\smallsetminus Z$, with Y smooth and $E=f^{-1}(Z)_{\rm red}$ a simple normal crossing divisor, then*

$$R^{r-1+i}f_*\Omega_Y^{n-i}(\log E)=0\quad for\quad i>n-\lceil\widetilde{\alpha}(Z)\rceil-1.$$

Outline of the paper. In the next section, we review some basic notions and results that we will need for the proofs of our main results. Theorem 1.1 and its corollaries, as well as Theorem 1.4 are proved in Section 3. Theorem 1.5 is proved in Section 4, while Theorem 1.7 is proved in Section 5.

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2. Background overview

In this section we recall some definitions and results that we will need. We work over the field $\mathbf{C}$ of complex numbers. By a variety we mean a reduced scheme of finite type over $\mathbf{C}$, not necessarily irreducible. For a variety Z, we denote by $Z_{\rm sing}$ the singular locus of Z.

2.1. **Mixed Hodge modules.** We only give a brief introduction to mixed Hodge modules and refer for proofs and details to [Sai90]. Let X be a smooth, irreducible, n-dimensional variety. We denote by $\mathcal{D}_X$ the sheaf of differential operators on X. For basic facts about $\mathcal{D}_X$-modules, we refer to [HTT08]. All the $\mathcal{D}_X$-modules we will consider will be left $\mathcal{D}_X$-modules. Since some of the results in the literature are stated for right $\mathcal{D}_X$-modules, we recall that there is an equivalence of categories between left and right $\mathcal{D}_X$-modules such that if $\mathcal{M}^r$ is the right $\mathcal{D}_X$-module corresponding to the left $\mathcal{D}_X$-module $\mathcal{M}$, then we have an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{M}^r\simeq\mathcal{M}\otimes_{\mathcal{O}_X}\omega_X.$$

When dealing with filtered $\mathcal{D}_X$-modules, the filtrations on $\mathcal{M}$ and $\mathcal{M}^r$ are indexed such that the above isomorphism maps $F_{p-n}\mathcal{M}^r$ to $F_p\mathcal{M}\otimes_{\mathcal{O}_X}\omega_X$ for all $p\in\mathbf{Z}$.

All filtrations on $\mathcal{D}_X$-modules that we will encounter are assumed to be good filtrations compatible with the filtration $F_\bullet\mathcal{D}_X$ on $\mathcal{D}_X$ by order of differential operators. This means that they are increasing, exhaustive filtrations by $\mathcal{O}_X$-submodules such that we have

$$F_p\mathcal{D}_X\cdot F_q\mathcal{M}\subseteq F_{p+q}\mathcal{M}\quad\text{for all}\quad p,q\in\mathbf{Z},$$

and there is q_0 such that this inclusion is an equality for all $p\geq 0$ and $q\geq q_0$. In this case we say that the filtration is *generated at level* q_0.

A mixed Hodge module $M=(\mathcal{M},F_\bullet\mathcal{M},\mathcal{P},\alpha,W_\bullet\mathcal{M})$ on X consists of several pieces of data: $\mathcal{M}$ is a $\mathcal{D}_X$-module on $\mathcal{M}$ (holonomic and with regular singularities), $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$ (the *Hodge filtration*), $W_\bullet\mathcal{M}$ is a finite increasing filtration on $\mathcal{M}$ by $\mathcal{D}_X$-submodules (the *weight filtration*), and $\mathcal{P}$ is a perverse sheaf over $\mathbf{Q}$ (sometimes written as ${\rm rat}(M)$), whose complexification is isomorphic via α to the perverse sheaf over $\mathbf{C}$ that corresponds to $\mathcal{M}$ via the Riemann-Hilbert correspondence. These data are supposed to satisfy a complicated set of conditions that we do not discuss. We refer to $(\mathcal{M}, F)$ as the filtered $\mathcal{D}_X$-module underlying M (though, with an abuse of notation, we sometimes write $F_k M$ and $W_k M$ instead of $F_k\mathcal{M}$ and $W_k\mathcal{M}$, respectively).

The Tate twist $M(k)$ of a mixed Hodge module M as above has the same underlying $\mathcal{D}_X$-module, but the two filtrations are shifted by

$$F_i\mathcal{M}(k) = F_{i-k}\mathcal{M} \quad \text{and} \quad W_i\mathcal{M}(k) = W_{i+2k}\mathcal{M} \quad \text{for all} \quad i \in \mathbf{Z}.$$

We note that the mixed Hodge modules on X form an Abelian category and every morphism of mixed Hodge modules is a morphism of $\mathcal{D}_X$-modules, which is strict with respect to both the Hodge and the weight filtration. There is a duality functor $\mathbf{D}$ on this category, lifting the usual duality functor on holonomic $\mathcal{D}_X$-modules. All our Hodge modules are polarizable, so the choice of a polarization implies that if M as above is *pure of weight* k (that is, $\mathrm{Gr}^W_i(M) = 0$ for $i \neq k$), we have an isomorphism $\mathbf{D}(M) \simeq M(k)$. For a general mixed Hodge module M and for every $k \in \mathbf{Z}$, the graded piece $\mathrm{Gr}^W_k(M)$, with the induced Hodge filtration, is a pure Hodge module of weight k.

An important example of mixed Hodge module (in fact, the only one that is easy to describe explicitly) is $\mathbf{Q}^H_X[n]$, which is a pure Hodge module of weight n. The underlying $\mathcal{D}_X$-module is $\mathcal{O}_X$ and the Hodge filtration is such that $\mathrm{Gr}^F_i(\mathcal{O}_X) = 0$ for all $i \neq 0$. The corresponding perverse sheaf is $\mathbf{Q}_X[n]$. Note that since $\mathbf{Q}^H_X[n]$ has weight n, a choice of polarization gives an isomorphism $\mathbf{D}\big(\mathbf{Q}^H_X[n]\big) \simeq \mathbf{Q}^H_X(n)[n]$.

Given a mixed Hodge module M, with underlying filtered $\mathcal{D}_X$-module $(\mathcal{M}, F)$, the Hodge filtration makes the de Rham complex of $\mathcal{M}$ a filtered complex. The graded pieces are, in fact, complexes of $\mathcal{O}_X$-modules. More precisely, $\mathrm{Gr}^F_p\mathrm{DR}_X(M)$ is the complex

$$0 \to \mathrm{Gr}^F_p(\mathcal{M}) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+1}(\mathcal{M}) \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+n}(\mathcal{M}) \to 0,$$

placed in cohomological degrees $-n, \ldots, 0$. For example, we have

$$\mathrm{Gr}^F_{-p}\mathrm{DR}_X\big(\mathbf{Q}^H_X[n]\big) = \Omega^p_Y[n-p].$$

We always think of $\mathrm{Gr}^F_p\mathrm{DR}_X(M)$ as an object in the derived category of coherent sheaves on X. This construction is compatible with proper push-forward (see [Sai88, Section 2.3.7]) and satisfies the following compatibility property with the duality functor by [Sai88, Sections 2.4.5 and 2.4.11]: for every p, we have a canonical isomorphism

$$\mathrm{Gr}^F_p\mathrm{DR}_X\big(\mathbf{D}(M)\big) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{-p}\mathrm{DR}_X(M), \omega_X[n]\big). \tag{1}$$

For future reference, we include the following lemma, in which we consider arbitrary filtered $\mathcal{D}_X$-modules:

Lemma 2.1. *If $f\colon (\mathcal{M}, F) \to (\mathcal{N}, F)$ is a morphism of filtered $\mathcal{D}_X$-modules on X and $k \in \mathbf{Z}$, then the induced morphism*

$$\mathrm{Gr}^F_p\mathrm{DR}_X(f)\colon \mathrm{Gr}^F_p\mathrm{DR}_X(M) \to \mathrm{Gr}^F_p\mathrm{DR}_X(N)$$

is an isomorphism (in the derived category) for all $p \leq k$ if and only if $F_p f\colon F_p M \to F_p N$ is an isomorphism for all $p \leq k+n$.

Proof. The "if" assertion follows directly from the definition of the graded de Rham complex. For the converse, arguing by induction, it is enough to show that if $F_p f$ is an isomorphism for all $p \leq k+n-1$ and $\mathrm{Gr}^F_p\mathrm{DR}_X(f)$ is an isomorphism (in the derived category), then $F_{k+n}(f)$

is an isomorphism. By hypothesis, we have a morphism of complexes placed in cohomological degrees $-n, \ldots, 0$:

$$\begin{array}{ccccccccc} 0 \to \mathrm{Gr}_k^F(M) & \to \cdots \to & \Omega_X^{n-1} \otimes_{\mathcal{O}_X} \mathrm{Gr}_{k+n-1}^F(M) & \to & \Omega_X^n \otimes_{\mathcal{O}_X} \mathrm{Gr}_{k+n}^F(M) \to 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ 0 \to \mathrm{Gr}_k^F(N) & \to \cdots \to & \Omega_X^{n-1} \otimes_{\mathcal{O}_X} \mathrm{Gr}_{k+n-1}^F(N) & \to & \Omega_X^n \otimes_{\mathcal{O}_X} \mathrm{Gr}_{k+n}^F(N) \to 0 \end{array}$$

such that the vertical maps in degrees $\neq 0$ are isomorphisms and such that the induced map for the 0-th cohomology is an isomorphism. An application of the 5-Lemma gives that the map $\mathrm{Gr}_{k+n}^F(f)$ is an isomorphism, and another application of the 5-Lemma implies that $F_{k+n}f$ is an isomorphism. □

One can define mixed Hodge modules also on a singular variety Z (if Z can be embedded in a smooth variety X, then we simply consider mixed Hodge modules on X whose support is contained in Z). One can consider the *derived category of mixed Hodge modules* on Z, denoted $D^b\big(\mathrm{MHM}(Z)\big)$. This satisfies a 6-functor formalism. For example, if $i\colon Z \hookrightarrow X$ is the inclusion, where X is smooth, then the underlying $\mathcal{D}_X$-module of the mixed Hodge module $\mathcal{H}^p\big(i_*i^!\mathbf{Q}_X^H[n]\big)$ is the local cohomology sheaf $\mathcal{H}_Z^p(\mathcal{O}_X)$ of $\mathcal{O}_X$ along Z.

For every variety Z, if $a_Z\colon Z \to \mathrm{pt}$ is the morphism to a point, then one defines $\mathbf{Q}_Z^H := a_Z^*(\mathbf{Q}_{\mathrm{pt}}^H)$ in $D^b\big(\mathrm{MHM}(Z)\big)$. If Z is smooth, then this coincides (up to a cohomological shift) with the object that we have already discussed. In general, however, it is a more complicated object. If X is a smooth, irreducible n-dimensional variety and $i\colon Z \hookrightarrow X$ is a closed embedding, then by functoriality we have a canonical isomorphism $\mathbf{Q}_Z^H \simeq i^*\mathbf{Q}_X^H$, so we have a canonical isomorphism

$$\mathbf{D}(\mathbf{Q}_Z^H) \simeq i^!\mathbf{Q}_X^H(n)[2n]. \tag{2}$$

For every Z, it is shown in [Sai90, Section 4.5] that $\mathbf{Q}_Z^H$ is of weight ≤ 0, that is, we have $\mathrm{Gr}_i^W\big(\mathcal{H}^j(\mathbf{Q}_Z^H)\big) = 0$ for $i > j$. Furthermore, if Z has pure dimension d, then $\mathcal{H}^i(\mathbf{Q}_Z^H) = 0$ for $i > d$ and the *intersection cohomology Hodge module*

$$\mathrm{IC}_Z\mathbf{Q}^H := \mathrm{Gr}_d^W\mathcal{H}^d(\mathbf{Q}_Z^H) \tag{3}$$

is the unique object of $\mathrm{MHM}(Z)$ whose restriction to $U = Z \smallsetminus Z_{\mathrm{sing}}$ is $\mathbf{Q}_U^H[d]$ and which has no subobject or quotient supported on Z_{sing}. The corresponding perverse sheaf is the intersection cohomology complex of Z; if Z is irreducible, then this is simple, hence so is $\mathrm{IC}_Z\mathbf{Q}^H$ and we have $\mathbf{Q} = \mathrm{End}\big(\mathrm{IC}_Z\mathbf{Q}^H\big)$. In general, if Z has N irreducible components, we have $\mathrm{End}\big(\mathrm{IC}_Z\mathbf{Q}^H\big) = \mathbf{Q}^N$ and a morphism $\big(\mathrm{IC}_Z\mathbf{Q}^H\big) \to \big(\mathrm{IC}_Z\mathbf{Q}^H\big)$ is uniquely determined by its restriction to the smooth locus of Z.

Note that by definition of $\mathrm{IC}_Z\mathbf{Q}^H$, we have a canonical morphism

$$\gamma_Z\colon \mathbf{Q}_Z^H[d] \to \mathrm{IC}_Z\mathbf{Q}^H. \tag{4}$$

Suppose now that X is a smooth, irreducible n-dimensional variety and $i\colon Z \hookrightarrow X$ is a closed embedding. Let $r = n - d$. Since $\mathrm{IC}_Z\mathbf{Q}^H = \mathrm{Gr}_d^W\mathcal{H}^d(\mathbf{Q}_Z^H)$ and $\mathrm{Gr}_p^W\mathcal{H}^d(\mathbf{Q}_Z^H) = 0$ for $p > d$, it follows using (2) that

$$i_*\mathbf{D}(\mathrm{IC}_Z\mathbf{Q}^H) \simeq \mathrm{Gr}_{-d}^W\big(i_*\mathcal{H}^{-d}\mathbf{D}(\mathbf{Q}_Z^H)\big) \simeq \mathrm{Gr}_{-d}^W\mathcal{H}^{-d}\big(i_*i^!\mathbf{Q}_X^H(n)[2n]\big) = \mathrm{Gr}_{n+r}^W\mathcal{H}_Z^r(\mathcal{O}_X)(n) \tag{5}$$

and $\mathrm{Gr}_p^W\mathcal{H}_Z^r(\mathcal{O}_X)=0$ for $p<n+r$. We note that this lowest weight piece of $\mathcal{H}_Z^r(\mathcal{O}_X)$ is the *intersection cohomology $\mathcal{D}$-module* introduced by Brylinski and Kashiwara in [BK81]; if Z is irreducible, then it can be characterized as the unique simple $\mathcal{D}_X$-submodule of $\mathcal{H}_Z^r(\mathcal{O}_X)$.

We also consider the shifted dual $\gamma_Z^\vee=\mathbf{D}(\gamma)(-d)$ of γ_Z, that can be identified via (2) to

$$\gamma_Z^\vee\colon \mathbf{D}(\mathrm{IC}_Z\mathbf{Q}^H)(-d)\to i^!\mathbf{Q}_X^H[n+r](n-d). \tag{6}$$

Note that since $\mathrm{IC}_Z\mathbf{Q}^H$ is pure of weight d, the choice of a polarization gives an isomorphism $\mathbf{D}(\mathrm{IC}_Z\mathbf{Q}^H)(-d)\simeq \mathrm{IC}_Z\mathbf{Q}^H$.

We will be especially interested in the case when Z is a locally complete intersection subvariety of X, of pure codimension r. In this case $\mathcal{H}_Z^i(\mathcal{O}_X)=0$ for all $i\neq r$, hence $i^!\mathbf{Q}_X^H[n+r]$ is a mixed Hodge module on Z. Duality implies that also $\mathbf{Q}_Z^H[d]$ is a mixed Hodge module on Z, hence γ_Z and $\gamma_Z^\vee$ are morphisms of mixed Hodge modules.

2.2. **V-filtrations.** Suppose that X is a smooth, irreducible, n-dimensional affine variety and $f_1,\ldots,f_r\in\mathcal{O}_X(X)=R$ are nonzero regular functions such that the ideal $(f_1,\ldots,f_r)$ defines the closed subscheme Z of X. We consider the graph embedding

$$\iota\colon X\hookrightarrow W=X\times\mathbf{A}^r,\quad \iota(x)=\big(x,f_1(x),\ldots,f_r(x)\big)$$

and the $\mathcal{D}$-module pushforward $B_{\mathbf{f}}=\iota_+\mathcal{O}_X$ (where $\mathbf{f}$ stands for $(f_1,\ldots,f_r)$). If $t_1,\ldots,t_r$ denote the standard coordinates on $\mathbf{A}^r$, then we can write

$$B_{\mathbf{f}}=\bigoplus_{\alpha\in\mathbf{Z}_{\geq 0}^r}R\partial_t^\alpha\delta_{\mathbf{f}},$$

where for $\alpha=(\alpha_1,\ldots,\alpha_r)$, we put $\partial_t^\alpha=\partial_{t_1}^{\alpha_1}\cdots\partial_{t_r}^{\alpha_r}$. The action of R and of ∂_{t_i} are the obvious ones, while the actions of $D\in\mathrm{Der}_{\mathbf{C}}(R)$ and of the t_i are given by

$$D\cdot h\partial_t^\alpha\delta_{\mathbf{f}}=D(h)\partial_t^\alpha\delta_{\mathbf{f}}-\sum_{i=1}^r D(f_i)h\partial_t^{\alpha+e_i}\delta_{\mathbf{f}}\quad\text{and}\quad t_i\cdot h\partial_t^\alpha\delta_{\mathbf{f}}=f_ih\partial_t^\alpha\delta_{\mathbf{f}}-\alpha_ih\partial_t^{\alpha-e_i}\delta_{\mathbf{f}},$$

where $e_1,\ldots,e_r$ is the standard basis of $\mathbf{Z}^d$. In fact, $B_{\mathbf{f}}$ underlies the pure Hodge module $\iota_*\mathbf{Q}_X^H[n]$, of weight n, with the Hodge filtration given by

$$F_{p+r}B_{\mathbf{f}}=\bigoplus_{|\alpha|\leq p}R\partial_t^\alpha\delta_{\mathbf{f}},$$

where for $\alpha=(\alpha_1,\ldots,\alpha_r)$, we put $|\alpha|=\alpha_1+\ldots+\alpha_r$.

The V-filtration on $B_{\mathbf{f}}$ has been constructed by Kashiwara [Kas83], extending work of Malgrange [Mal83] in the case $r=1$. It is a decreasing, exhaustive filtration indexed by rational numbers $(V^\lambda B_{\mathbf{f}})_{\lambda\in\mathbf{Q}}$. It is discrete and left-continuous and it is characterized by several properties, the most important of these saying that for every $\lambda\in\mathbf{Q}$

$$t_i\cdot V^\lambda B_{\mathbf{f}}\subseteq V^{\lambda+1}B_{\mathbf{f}}\quad\text{and}\quad \partial_{t_i}\cdot V^\lambda B_{\mathbf{f}}\subseteq V^{\lambda-1}B_{\mathbf{f}},$$

and if $s=-\sum_{i=1}^r\partial_{t_i}t_i$, then $s+\lambda$ is nilpotent on $\mathrm{Gr}_V^\lambda(B_{\mathbf{f}})=V^\lambda B_{\mathbf{f}}/V^{>\lambda}B_{\mathbf{f}}$, where $V^{>\lambda}B_{\mathbf{f}}=\bigcup_{\beta>\lambda}V^\beta B_{\mathbf{f}}$. Note that the Hodge filtration on $B_{\mathbf{f}}$ induces a Hodge filtration on each $\mathrm{Gr}_V^\lambda(B_{\mathbf{f}})$.

In fact, a V-filtration exists on $\iota_+\mathcal{M}$, whenever $\mathcal{M}$ underlies a mixed Hodge module. In the case $r=1$, the interplay between the Hodge filtration on $\mathcal{M}$ and V-filtrations plays an important role in the definition of mixed Hodge modules. For details about the construction and properties of V-filtrations, see [BMS06].

Let $i\colon Z\hookrightarrow X$ be the inclusion. For $r=1$, the V-filtration is the key ingredient for the definition of $i^!(M)$ and $i^*(M)$ when M is a mixed Hodge module on X. In the case $r>1$, the corresponding description does not follow from the definition of these functors, but it has been recently proved in [CD21, Theorem 1.2]. We only state this in the case $M=\mathbf{Q}_X^H[n]$.

Theorem 2.2. *With the above notation, the following hold: the complex*

$$0\to \mathrm{Gr}_V^0(B_{\mathbf{f}})(-r)\xrightarrow{(t_1,t_2,\ldots,t_r)}\bigoplus_{i=1}^r \mathrm{Gr}_V^1(B_{\mathbf{f}})(-r)\to\cdots\to \mathrm{Gr}_V^r(B_{\mathbf{f}})(-r)\to 0,$$

placed in cohomological degrees $0,\ldots,r$ *represents* $i_*i^!\mathbf{Q}_X^H[n]$ *in the derived category of filtered* $\mathcal{D}_X$*-modules and the complex*

$$0\to \mathrm{Gr}_V^r(B_{\mathbf{f}})\xrightarrow{(\partial_{t_1},\ldots,\partial_{t_r})}\bigoplus_{i=1}^r \mathrm{Gr}_V^{r-1}(B_{\mathbf{f}})(-1)\to\cdots\to \mathrm{Gr}_V^0(B_{\mathbf{f}})(-r)\to 0$$

placed in cohomological degrees $-r,\ldots,0$ *represents* $i_*i^*\mathbf{Q}_X^H[n]$.

2.3. **The minimal exponent.** We next discuss the minimal exponent for locally complete intersection varieties, following [CDMO22]. Let X be a smooth, irreducible, n-dimensional variety and Z a (nonempty) closed subscheme of X, which is locally a complete intersection of pure codimension r. Suppose first that $X=\mathrm{Spec}(R)$ is affine and Z is defined by the ideal generated by $f_1,\ldots,f_r\in R$. The minimal exponent $\widetilde{\alpha}(Z)$ is defined by[1]

$$\widetilde{\alpha}(Z)=\begin{cases}\sup\{\gamma>0\mid \delta_{\mathbf{f}}\in V^\gamma B_{\mathbf{f}}\}, & \text{if } \delta_{\mathbf{f}}\notin V^rB_{\mathbf{f}};\\ \sup\{r-1+q+\gamma\mid F_{q+r}B_{\mathbf{f}}\subseteq V^{r-1+\gamma}B_{\mathbf{f}}\}, & \text{if } \delta_{\mathbf{f}}\in V^rB_{\mathbf{f}}.\end{cases}\tag{7}$$

In general, we consider a cover $X=U_1\cup\ldots\cup U_N$, where each U_i is an affine open subset as above, and put

$$\widetilde{\alpha}(Z)=\min_{i;Z\cap U_i\neq\emptyset}\widetilde{\alpha}(Z\cap U_i).$$

It follows from [BMS06, Theorem 1] that we always have $\min\big\{\widetilde{\alpha}(Z),r\big\}=\mathrm{lct}(X,Z)$, the *log canonical threshold* of the pair (X,Z). Therefore the minimal exponent is interesting precisely when $\mathrm{lct}(X,Z)=r$, in which case Z is automatically reduced (see [CDMO22, Remark 4.2]). Moreover, it follows from [CDMO22, Corollary 1.7] that Z has rational singularities if and only if $\widetilde{\alpha}(Z)>r$. One can also show that Z is smooth if and only if $\widetilde{\alpha}(Z)=\infty$; in fact, if $x\in Z$ is a singular point, then we have the following more precise bound (see [CDMO22, Remark 4.21]):

$$\widetilde{\alpha}(Z)\leq n-\tfrac{1}{2}\dim_{\mathbf{C}}T_xZ.\tag{8}$$

The minimal exponent $\widetilde{\alpha}(Z)$ depends on the ambient variety X, but in a predictable way: the difference $\widetilde{\alpha}(Z)-\dim(X)$ only depends on Z (see [CDMO22, Proposition 4.14]).

When $r=1$, the minimal exponent was defined by Saito [Sai94] as the negative of the largest root of the *reduced Bernstein-Sato polynomial* $\widetilde{b}_Z(s)$. The fact that this agrees with the above definition is a consequence of [Sai16, (1.3.8)].

Recall now that the $\mathcal{D}_X$-module $\mathcal{H}_Z^r(\mathcal{O}_X)$ underlies a mixed Hodge module on X, namely $\mathcal{H}^r\big(i_*i^!\mathbf{Q}_X^H[n]\big)$, where $i\colon Z\hookrightarrow X$ is the inclusion. We thus have a canonical filtration on

[1] We note that what we denote by $F_{p+r}B_{\mathbf{f}}$ here is denoted by $F_pB_{\mathbf{f}}$ in [CDMO22].

$\mathcal{H}^r_Z(\mathcal{O}_X)$, the Hodge filtration $\big(F_p\mathcal{H}^r_Z(\mathcal{O}_X)\big)_{p\geq 0}$. We have a second filtration, the *order* filtration $\big(E_p\mathcal{H}^r_Z(\mathcal{O}_X)\big)_{p\geq 0}$, given by

$$E_p\mathcal{H}^r_Z(\mathcal{O}_X)=\big\{u\in\mathcal{H}^r_Z(\mathcal{O}_X)\mid I_Z^{p+1}u=0\big\}=\mathrm{Im}\big(\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^{p+1},\mathcal{O}_X)\hookrightarrow\mathcal{H}^r_Z(\mathcal{O}_X)\big),$$

where I_Z is the ideal defining Z in X (see [MP22a, Proposition 3.11]). It is a general fact that $F_p\mathcal{H}^r_Z(\mathcal{O}_X)\subseteq E_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for all $p\geq 0$ (see [MP22a, Proposition 3.4]) and the following result shows that the minimal exponent governs how far these two filtrations agree (see [CDMO22, Theorem 1.3]):

Theorem 2.3. *If X is a smooth, irreducible variety and Z is a locally complete intersection subvariety of pure codimension r in X, then for a nonnegative integer k, we have $F_p\mathcal{H}^r_Z(\mathcal{O}_X)=E_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for $0\leq p\leq k$ if and only if $\widetilde{\alpha}(Z)\geq r+k$.*

2.4. **k-Du Bois singularities.** To a variety Z, Du Bois associated in [DB81] a complex $\underline{\Omega}^\bullet_Z$, known now as the *Du Bois complex* of Z. This is a filtered complex that agrees with the de Rham complex $\Omega^\bullet_Z$, with the "stupid" filtration, when Z is smooth. This allows extending to singular varieties some important cohomological properties of the de Rham complex of smooth varieties, see [PS08, Chapter 7.3] for an introduction to this topic.

We are interested in the shifted truncations $\underline{\Omega}^p_Z:=\mathrm{Gr}^p_F(\underline{\Omega}^\bullet_Z)[p]$, which are objects in the bounded derived category $D^b_{\mathrm{coh}}(Z)$ of coherent sheaves on Z. For every p, there is a canonical morphism $\Omega^p_Z\to\underline{\Omega}^p_Z$ that is an isomorphism over the smooth locus of Z. Following [JKSY21], we say that Z has k-Du Bois singularities, for some nonnegative integer k, if these morphisms are isomorphisms for all $0\leq p\leq k$. Note that for $k=0$, we recover the familiar notion of *Du Bois singularities*.

As we have mentioned in the Introduction, it was shown in [MP22a, Theorem F] that if X is a smooth, irreducible variety and Z is a locally complete intersection subvariety of X, of pure codimension r, then Z has k-Du Bois singularities if and only if $F_p\mathcal{H}^r_Z(\mathcal{O}_X)=E_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for $p\leq k$. In terms of minimal exponents, this condition can be rephrased as $\widetilde{\alpha}(Z)\geq r+k$. The proof of this result in *loc. cit.* extends the argument in the case of hypersurfaces, for which the two implications had previously been proved in [MOPW21] and [JKSY21].

Remark 2.4. If Z is a locally complete intersection variety with k-Du Bois singularities, then $\mathrm{codim}_Z(Z_{\mathrm{sing}})\geq 2k+1$. Indeed, this is a local statement, hence we may assume that Z has pure dimension (we use the fact that Z is Cohen-Macaulay) and that it is a closed subvariety of the smooth irreducible variety X. In this case the assertion follows by combining [MP22a, Corollary 3.40 and Theorem F].

The connection between the Du Bois complex and mixed Hodge modules is provided by the following result of Saito. If Z is a closed subvariety of the smooth, irreducible, n-dimensional variety X and $i\colon Z\hookrightarrow X$ is the inclusion, then it is a consequence of [Sai00, Theorem 4.2] that for every p, we have an isomorphism

$$\underline{\Omega}^p_Z[-p]\simeq\mathrm{Gr}^F_{-p}\mathrm{DR}_X(i_*\mathbf{Q}^H_Z) \tag{9}$$

in $D^b_{\mathrm{coh}}(X)$. In light of (1) and (2), this is equivalent to

$$\underline{\Omega}^p_Z[-p]\simeq\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_Xi_*i^!\mathbf{Q}^H_X[n],\omega_X\big). \tag{10}$$

For an easy proof of this isomorphism, see [MP22a, Proposition 5.5].

2.5. **k-rational singularities.** Given a variety Z, by a *strong log resolution* of Z we mean a proper morphism $\mu\colon \widetilde{Z} \to Z$ that is an isomorphism over $Z \smallsetminus Z_{\rm sing}$, such that $\widetilde{Z}$ is smooth and $E = \mu^{-1}(Z_{\rm sing})$ is a simple normal crossing divisor. For a nonnegative integer k, following [FL22a], we say that Z has *k-rational* singularities if the canonical morphism

$$\Omega_Z^p \to \mathbf{R}\mu_*\Omega^p_{\widetilde{Z}}(\log E) \tag{11}$$

is an isomorphism for all $p \leq k$. This is easily seen to be independent of the log resolution (see for example [MP22b, Lemma 1.6]). Note that for $k = 0$ we recover the classical notion of rational singularities. This condition implies that Z is normal, hence in particular, every connected component of Z is irreducible. The notion of k-rational singularities has been extensively studied in [FL22a], [FL22b], [FL22c], [MP22b].

For our purpose it will be convenient to consider a different description of k-rational singularities. Recall from [MP22b, Section 6] that for every variety Z of pure dimension d and every nonnegative integer k, we have a canonical morphism

$$\psi_k\colon \underline{\Omega}_Z^k \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}\big(\underline{\Omega}_Z^{d-k}, \omega_Z^\bullet[-d]\big), \tag{12}$$

where $\omega_Z^\bullet$ is the dualizing complex of Z. This is defined as follows: suppose that $\mu\colon Y \to Z$ is an arbitrary resolution of singularities (we only require that Y is smooth and μ is proper and an isomorphism over a dense open subset of Z). By functoriality of the Du Bois complex, for every nonnegative integer k, we have a canonical morphism $\alpha_k\colon \underline{\Omega}_Z^k \to \mathbf{R}\mu_*\Omega_Y^k$. On the other hand, on Y we have a canonical isomorphism

$$\Omega_Y^k \xrightarrow{\simeq} \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\Omega_Y^{d-k}, \omega_Y^\bullet[-d]\big).$$

By pushing this forward and using Grothendieck duality for μ, we obtain an isomorphism β_k as the composition

$$\mathbf{R}\mu_*\Omega_Y^k \xrightarrow{\simeq} \mathbf{R}\mu_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\Omega_Y^{d-k}, \omega_Y^\bullet[-d]\big) \xrightarrow{\simeq} \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}\big(\mathbf{R}\mu_*\Omega_Y^{d-k}, \omega_Z^\bullet[-d]\big).$$

The morphism ψ_k is obtained as the composition $\alpha_{d-k}^\vee \circ \beta_k \circ \alpha_k$, where we put $\alpha_{d-k}^\vee = \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}\big(\alpha_{d-k}, \omega_Z^\bullet[-d]\big)$. It is shown in [MP22b, Proposition 6.1] that this definition does not depend on the choice of resolution of singularities.

With this notation, we have the following characterization of k-rational singularities in the locally complete intersection case, see [MP22b] (in *loc. cit.* one assumes that Z is irreducible, but the argument works in general):

Theorem 2.5. *If Z is a locally complete intersection variety of pure dimension d and k is a nonnegative integer, then Z has k-rational singularities if and only if Z has k-Du Bois singularities and the morphism $\psi_k\colon \underline{\Omega}_Z^k \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}\big(\underline{\Omega}_Z^{d-k}, \omega_Z\big)$ is an isomorphism.*

It will be important for us to use an interpretation of the morphism ψ_k from [FL22b, Appendix], as the graded de Rham of a morphism of mixed Hodge modules. Let Z be a variety of pure dimension d and $\mu\colon Y \to Z$ any resolution of singularities, with μ a projective morphism. Note that by functoriality we have a canonical morphism of mixed Hodge modules $\alpha\colon \mathbf{Q}_Z^H[d] \to \mu_*\mathbf{Q}_Y^H[d]$. On the other hand, since $\mathbf{Q}_Y^H[d]$ is pure of weight d, on Y we have a canonical isomorphism $\mathbf{Q}_Y^H[d] \to \mathbf{D}\big(\mathbf{Q}_Y^H[d]\big)(-d)$, which after pushing forward to Z and using the compatibility of pushforward with duality, gives an isomorphism

$$\beta\colon \mu_*\mathbf{Q}_Y^H[d] \xrightarrow{\simeq} \mu_*\mathbf{D}\big(\mathbf{Q}_Y^H[d]\big)(-d) \xrightarrow{\simeq} \mathbf{D}\big(\mu_*\mathbf{Q}_Y^H[d]\big)(-d).$$

We then obtain a morphism ψ_Z in the derived category of mixed Hodge modules on Z as the following composition

$$\mathbf{Q}_Z^H[d] \xrightarrow{\alpha} \mu_*\mathbf{Q}_Y^H[d] \xrightarrow{\beta} \mathbf{D}\big(\mu_*\mathbf{Q}_Y^H[d]\big)(-d) \xrightarrow{\alpha^\vee} \mathbf{D}\big(\mathbf{Q}_Z^H[d]\big)(-d), \tag{13}$$

where $\alpha^\vee = \mathbf{D}(\alpha)(-d)$.

If Z is a closed subvariety of the smooth, irreducible variety X and $i\colon Z \hookrightarrow X$ is the inclusion, then using the compatibility of the graded de Rham complex with direct image and duality, we see that for every $k \in \mathbf{Z}$ we have

$$\alpha_k = \mathrm{Gr}^F_{-k}\mathrm{DR}_X(i_*\alpha)[k-d],\ \beta_k = \mathrm{Gr}^F_{-k}\mathrm{DR}_X(i_*\beta)[k-d],\ \alpha^\vee_{d-k} = \mathrm{Gr}^F_{-k}\mathrm{DR}_X(i_*\alpha^\vee)[k-d],$$

hence $\psi_k = \mathrm{Gr}^F_{-k}\mathrm{DR}_X(i_*\psi_Z)[k-d]$.

Remark 2.6. It follows from the definition of ψ_Z that $\psi_Z^\vee := \mathbf{D}(\psi_Z)(-d)$ can be identified with ψ_Z.

Remark 2.7. Suppose now that X is a smooth, irreducible, n-dimensional variety and $i\colon Z \hookrightarrow X$ is a closed embedding, where Z is a locally complete intersection subvariety of X, of pure codimension r. Let $d = n - r$. As we have already mentioned, in this case, the morphisms

$$\mathbf{Q}_Z^H[d] \xrightarrow{\gamma_Z} \mathrm{IC}_Z\mathbf{Q}^H \quad \text{and} \quad \mathbf{D}\big(\mathrm{IC}_Z\mathbf{Q}^H\big)(-d) \xrightarrow{\gamma_Z^\vee} \mathbf{D}\big(\mathbf{Q}_Z^H[d]\big)(-d)$$

are morphisms of mixed Hodge modules, with γ_Z surjective and $\gamma_Z^\vee$ injective.

Since ψ_Z is a morphism between two mixed Hodge modules on Z, we obtain the same morphism if we take $\mathcal{H}^0(-)$; in other words, ψ_Z agrees with the composition

$$\mathbf{Q}_Z^H[d] \to \mathcal{H}^0\big(\mu_*\mathbf{Q}_Y^H[d]\big) \to \mathbf{D}\big(\mathcal{H}^0(\mu_*\mathbf{Q}_Y^H[d])\big)(-d) \to \mathbf{D}\big(\mathbf{Q}_Z^H[d]\big)(-d). \tag{14}$$

On the other hand, since $\mathbf{Q}_Y^H[d]$ is pure of weight d, so is $\mathcal{H}^0\big(\mu_*\mathbf{Q}_Y^H[d]\big)$, see [Sai88, Théorème 1]. Since $\mathrm{Gr}^W_i\big(\mathbf{Q}_Z^H[d]\big) = 0$ for $i > d$, it follows that the composition in (14) further factors as

$$\mathbf{Q}_Z^H[d] \xrightarrow{\gamma_Z} \mathrm{IC}_Z\mathbf{Q}^H \to \mathcal{H}^0\big(\mu_*\mathbf{Q}_Y^H[d]\big) \to \mathbf{D}\big(\mathcal{H}^0(\mu_*\mathbf{Q}_Y^H[d])\big)(-d)$$
$$\to \mathbf{D}\big(\mathrm{IC}_Z\mathbf{Q}^H\big)(-d) \xrightarrow{\gamma_Z^\vee} \mathbf{D}\big(\mathbf{Q}_Z^H[d]\big)(-d). \tag{15}$$

We also note that the intermediate composition

$$\mathrm{IC}_Z\mathbf{Q}^H \to \mathcal{H}^0\big(\mu_*\mathbf{Q}_Y^H[d]\big) \to \mathbf{D}\big(\mathcal{H}^0(\mu_*\mathbf{Q}_Y^H[d])\big)(-d) \to \mathbf{D}\big(\mathrm{IC}_Z\mathbf{Q}^H\big)(-d)$$

is always an isomorphism. Indeed, a morphism $\mathrm{IC}_Z\mathbf{Q}^H \to \mathbf{D}\big(\mathrm{IC}_Z\mathbf{Q}^H\big)(-d)$ is uniquely determined by its restriction to a dense open subset of the smooth locus of Z, and on a suitable such subset over which μ is an isomorphism this composition is the identity.

For every k, it follows from Lemma 2.1 that $\mathrm{Gr}^F_p\mathrm{DR}_X(i_*\psi_Z)$ is an isomorphism for all $p \le k$ if and only if $F_p i_*\psi_Z$ is an isomorphism for every $p \le k+n$. Since $i_*\gamma_Z$ is surjective and $i_*\gamma_Z^\vee$ is injective, it follows from the above discussion that $\mathrm{Gr}^F_p\mathrm{DR}_X(i_*\psi_Z)$ is an isomorphism for all $p \le k$ if and only if

$$F_p i_*\gamma_Z\colon F_p i_*\mathbf{Q}_Z^H[d] \to F_p i_*\mathrm{IC}_Z\mathbf{Q}^H \quad \text{and}$$

$$F_{p+d} i_*\gamma_Z^\vee\colon F_{p+d}\mathbf{D}(i_*\mathrm{IC}_Z\mathbf{Q}^H) = F_{p-r}W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) \to F_{p+d} i_*\mathbf{D}\big(\mathbf{Q}_Z^H[d]\big) = F_{p-r}\mathcal{H}^r_Z(\mathcal{O}_X)$$

are isomorphisms for all $p \le k+n$.

Remark 2.8. We note that in [FL22b] one says that a variety Z of pure dimension d has k-rational singularities if the composition

$$\Omega_Z^p \to \underline{\Omega}_Z^p \xrightarrow{\psi_k} \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}\big(\underline{\Omega}_Z^{d-k}, \omega_Z^\bullet[-d]\big)$$

is an isomorphism for all $p \leq k$. It is shown in [FL22b, Corollary 3.17] that this definition is equivalent to the definition we use in this paper if $\mathrm{codim}_Z(Z_{\mathrm{sing}}) \geq 2k+1$. Furthermore, it is shown in [FL22b, Theorem 3.20] that with their definition as well, if Z is a locally complete intersection and has k-rational singularities, then Z has Du Bois singularities, and thus $\mathrm{codim}_Z(Z_{\mathrm{sing}}) \geq 2k+1$ by Remark 2.4. We thus conclude that for locally complete intersection varieties, the two definitions of k-rational singularities agree.

3. Characterizations of k-rationality for locally complete intersections

Let X be a smooth, irreducible variety of dimension n and Z be a locally complete intersection subvariety of pure codimension r in X. Let $d = n - r$ be the dimension of Z and $i\colon Z \hookrightarrow X$ the inclusion. We will freely use the notation introduced in the previous section. The following is the main result of this section, which implies several of the statements in the introduction.

Theorem 3.1. *With the above notation, for every nonnegative integer k, the following assertions are equivalent:*

(a) $\widetilde{\alpha}(Z) > k + r$;

(b) $F_k W_{n+r}\mathcal{H}_Z^r(\mathcal{O}_X) = E_k\mathcal{H}_Z^r(\mathcal{O}_X)$;

(c) *The morphism*

$$F_{p+r}i_*\mathbf{Q}_Z^H[d] \twoheadrightarrow F_{p+r}i_*\mathrm{IC}_Z\mathbf{Q}^H, \tag{16}$$

induced by γ_Z and the composition

$$F_pW_{n+r}\mathcal{H}_Z^r(\mathcal{O}_X) \hookrightarrow F_p\mathcal{H}_Z^r(\mathcal{O}_X) \hookrightarrow E_p\mathcal{H}_Z^r(\mathcal{O}_X), \tag{17}$$

induced by $\gamma_Z^\vee$, are isomorphisms for $p \leq k$.

(d) *Z has k-Du Bois singularities and the morphism*

$$\psi_k\colon \underline{\Omega}_Z^k \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}\big(\underline{\Omega}_Z^{d-k}, \omega_Z\big)$$

is an isomorphism;

(e) *the canonical morphism*

$$\Omega_Z^p \to \mathrm{Gr}_{-p}^F\mathrm{DR}_X(i_*\mathrm{IC}_Z\mathbf{Q}^H)[p-d]$$

is an isomorphism (in the derived category) for $p \leq k$.

Remark 3.2. We note that the morphism in (e) is the composition

$$\Omega_Z^p \to \underline{\Omega}_Z^p \simeq \mathrm{Gr}_{-p}^F\mathrm{DR}_X(i_*\mathbf{Q}_Z^H)[p] \to \mathrm{Gr}_{-p}^F\mathrm{DR}_X(i_*\mathrm{IC}_Z\mathbf{Q}^H)[p-d],$$

where the second map is induced by $\gamma_Z\colon \mathbf{Q}_Z^H[d] \to \mathrm{IC}_Z\mathbf{Q}^H$.

Remark 3.3. Note that the assertion (d) in the theorem is equivalent to the fact that Z has k-rational singularities by Theorem 2.5. Therefore the equivalence (a)⇔(d) is the content of Theorem 1.1, while the equivalence (a)⇔(b) is the content of Theorem 1.4.

We proceed with the proof of Theorem 3.1 in several steps, by showing the following implications:

$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) + (e)$, $(d) \Rightarrow (c)$, and $(e) \Rightarrow (b) \Rightarrow (a)$.

Proof of Theorem 3.1. Since all assertions are local, we may and will assume that X is affine and Z is defined in X by $f_1, \ldots, f_r \in \mathcal{O}_X(X)$. In particular, we will be able to consider the V-filtration corresponding to these functions. We denote by I_Z the ideal defining Z in X.

Step 1. Proof of (a)⇒(b). Let $W_\bullet$ be the monodromy filtration on $\mathrm{Gr}_V^\bullet B_{\mathbf{f}}$, uniquely characterized by:

- $(s+\alpha) \cdot W_\bullet \mathrm{Gr}_V^\alpha B_{\mathbf{f}} \subseteq W_{\bullet-2}\mathrm{Gr}_V^\alpha B_{\mathbf{f}}$ and
- $(s+\alpha)^j \colon \mathrm{Gr}^W_{n+j}\mathrm{Gr}_V^\alpha B_{\mathbf{f}} \cong \mathrm{Gr}^W_{n-j}\mathrm{Gr}_V^\alpha B_{\mathbf{f}}$ is an isomorphism for all $j \geq 1$.

Explicitly, this is given by

$$W_{n+i}\mathrm{Gr}_V^\alpha B_{\mathbf{f}} = \sum_j \ker\big((s+\alpha)^{i+j+1}\big) \cap \mathrm{Im}\big((s+\alpha)^j\big) \tag{18}$$

Consider the map $\sigma \colon \big(\mathrm{Gr}_V^{r-1} B_{\mathbf{f}}\big)^{\oplus r} \xrightarrow{(t_1, t_2, \ldots, t_r)} \mathrm{Gr}_V^r B_{\mathbf{f}}$. By [CD21, Theorem 1.2], for every i, we have an isomorphism of filtered $\mathcal{D}_X$-modules

$$\mathrm{Gr}^W_{i+r}\mathcal{H}^r_Z(\mathcal{O}_X) \simeq \big(\mathrm{Gr}^W_i \operatorname{coker}\sigma, F[-r]\big) \quad \text{for all} \quad i \in \mathbf{Z}. \tag{19}$$

Recall that we know that $W_{i+r}\mathcal{H}^r_Z(\mathcal{O}_X) = 0$ for $i < n$, hence

$$W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = \mathrm{Gr}^W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) \simeq \big(\mathrm{Gr}^W_n \operatorname{coker}\sigma, F[-r]\big). \tag{20}$$

Since $\widetilde{\alpha}(Z) > k+r$, it follows from the definition of the minimal exponent that $F_{k+r+1}B_{\mathbf{f}} \subseteq V^{>r-1}B_{\mathbf{f}}$, hence

$$(s+r) \cdot F_{k+r}B_{\mathbf{f}} \subseteq \sum_{i=1}^r t_i \cdot F_{k+r+1}B_{\mathbf{f}} \subseteq V^{>r}B_{\mathbf{f}}.$$

We thus have

$$F_{k+r}\mathrm{Gr}_V^r B_{\mathbf{f}} \subseteq W_n \mathrm{Gr}_V^r B_{\mathbf{f}} \tag{21}$$

because $\ker(s+r) \subseteq W_n\mathrm{Gr}_V^r B_{\mathbf{f}}$ by (18). This implies that $F_{k+r}\operatorname{coker}\sigma \subseteq W_n \operatorname{coker}\sigma$, and using (20) we conclude that

$$F_k W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = F_k\mathcal{H}^r_Z(\mathcal{O}_X) = E_k\mathcal{H}^r_Z(\mathcal{O}_X),$$

where the last equality follows from Theorem 2.3.

Step 2. Proof of (b)⇒(c). We first prove the following

Lemma 3.4. *The equality $F_kW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = E_k\mathcal{H}^r_Z(\mathcal{O}_X)$ implies*

$$F_pW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = E_p\mathcal{H}^r_Z(\mathcal{O}_X) \quad \textit{for all} \quad p \leq k.$$

Proof. Since $W_{n+r}\mathcal{H}^r(Z)$ is a Hodge module supported on Z, we have

$$I_Z \cdot F_pW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) \subseteq F_{p-1}W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) \quad \text{for all} \quad p \in \mathbf{Z}$$

(see [Sai88, Lemme 3.2.6]). On the other hand, it follows easily from the definition of the filtration $E_\bullet\mathcal{H}^r_Z(\mathcal{O}_X)$ that we have

$$I_Z \cdot E_p\mathcal{H}^r_Z(\mathcal{O}_X) = E_{p-1}\mathcal{H}^r_Z(\mathcal{O}_X) \quad \text{for all} \quad p \geq 1.$$

The assertion in the lemma now follows by decreasing induction on p. □

We next use duality to prove the following

Lemma 3.5. *If $F_pW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = F_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for some $p \in \mathbf{Z}$, then the surjective map*

$$F_{p+r+1}i_*\mathbf{Q}^H_Z[d] \to F_{p+r+1}i_*\mathrm{IC}_Z\mathbf{Q}^H$$

induced by γ_Z is an isomorphism.

Proof. The equality $F_pW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = F_p\mathcal{H}^r_Z(\mathcal{O}_X)$ is equivalent to $F_p\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X) = 0$ for all $j \geq 1$. Since $\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X)$ underlies a polarizable pure Hodge module of weight $n+r+j$, the choice of a polarization gives an isomorphism of filtered $\mathcal{D}_X$-modules:

$$\mathbf{D}\big(\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X)\big) \xrightarrow{\simeq} \big(\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X)\big)(n+r+j). \tag{22}$$

On the other hand,

$$\begin{aligned}\mathbf{D}\big(\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X)\big) &\cong \mathrm{Gr}^W_{-n-r-j}\mathbf{D}\big(\mathcal{H}^r_Z(\mathcal{O}_X)\big)\\ &\cong \mathrm{Gr}^W_{-n-r-j}\big(i_*\mathbf{Q}^H_Z[d](n)\big) \cong \big(\mathrm{Gr}^W_{d-j}i_*\mathbf{Q}^H_Z[d]\big)(n).\end{aligned} \tag{23}$$

Combining the two equations (22) and (23) yields

$$\big(\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X)\big)(r+j) \cong \mathrm{Gr}^W_{d-j}i_*\mathbf{Q}^H_Z[d],$$

as filtered $\mathcal{D}_X$-modules, which implies

$$F_{p+1-j}\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X) \cong F_{p+r+1}\mathrm{Gr}^W_{d-j}i_*\mathbf{Q}^H_Z[d].$$

We have seen that our hypothesis gives $F_{p+1-j}\mathrm{Gr}^W_{n+r+j}\mathcal{H}^r_Z(\mathcal{O}_X) = 0$ for $j \geq 1$, hence $F_{p+r+1}\mathrm{Gr}^W_{d-j}i_*\mathbf{Q}^H_Z[d] = 0$ for $j \geq 1$ and thus $F_{p+r+1}W_{d-1}i_*\mathbf{Q}^H_Z[d] = 0$. This implies the conclusion of the lemma by definition of γ_Z. □

Returning to the proof of the implication (b)⇒(c), note that the assertion in Lemma 3.4 gives the fact that the morphism (17) is an isomorphism for $p \leq k$. Similarly, by combining Lemmas 3.4 and 3.5 we conclude that the morphism (16) is an isomorphism for $p \leq k$ (in fact, for $p \leq k+1$). We thus have the assertion in (c).

Step 3. Proof of (c)⇒(d)+(e). The surjectivity of the morphism in (17) implies, in particular, that Z is k-Du Bois by [MP22a, Theorem F]. Since $\psi_k = \mathrm{Gr}^F_{-k}\mathrm{DR}_X(i_*\psi_Z)[k-d]$, we see that ψ_k is an isomorphism if and only if $\mathrm{Gr}^F_{-k}\mathrm{DR}_X(i_*\psi_Z)$ is an isomorphism, which by (1) holds if and only if $\mathrm{Gr}^F_{k-d}\mathrm{DR}_X(i_*\psi_Z)$ is an isomorphism (recall that $\mathbf{D}(\psi_Z) = \psi_Z(d)$, see Remark 2.6). We thus get assertion in (d).

Similarly, once we know that Z is k-Du Bois, the assertion in (e) is equivalent with the fact that $\mathrm{Gr}^F_{-p}\mathrm{DR}_X(i_*\gamma_Z)$ is an isomorphism for $p \leq k$, which is equivalent by (1) with the fact that $\mathrm{Gr}^F_{p-d}\mathrm{DR}_X(i_*\gamma_Z^\vee)$ is an isomorphism for all $p \leq k$. This is implied by $F_{p+r}i_*\gamma_Z^\vee$ being an isomorphism for all $p \leq k$, but this is precisely the morphism $F_pW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) \to F_p\mathcal{H}^r_Z(\mathcal{O}_X)$. Therefore we have the assertion in (e) as well.

Step 4. Proof of (d)⇒(c). It follows from Theorem 2.5 that the conditions in (d) are equivalent to Z having k-rational singularities. In particular, since we have these conditions for k, we also have them for $k-1$. In particular, we know that $\psi_k = \mathrm{Gr}^F_{p-d}\mathrm{DR}_X(i_*\psi_Z)[p-d]$ is an isomorphism for all $p \leq k$. Using (1) and the fact that $\mathbf{D}(\psi_Z) = \psi_Z(d)$, we conclude that $\mathrm{Gr}^F_{p-d}\mathrm{DR}_X(i_*\psi_Z)$ is an isomorphism for all $p \leq k$. Lemma 2.1 thus implies that $F_{p+r}i_*\psi_Z$ is an isomorphism for all $p \leq k$. As we have seen in Remark 2.7, this implies that the morphisms (16) and (17) are isomorphisms for all $p \leq k$ (for the latter morphism, we also use the fact that $F_p\mathcal{H}^r_Z(\mathcal{O}_X)=E_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for $p\leq k$, due to the fact that Z has k-Du Bois singularities). We thus have the condition in (c).

Step 5. Proof of (e)⇒(b). If $k=0$, applying $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(-,\omega_X[r]\big)$ to the isomorphism in (e) gives via (1) an isomorphism

$$\mathrm{Gr}^F_{-n}\mathrm{DR}_X W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)\to \mathrm{Gr}^F_{-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X)\to \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z,\omega_X)$$

(note that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Z,\omega_X)=0$ for $i\neq r$ since Z is a locally complete intersection of pure codimension r). We have

$$\mathrm{Gr}^F_{-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X)=\omega_X\otimes_{\mathcal{O}_X}F_0\mathcal{H}^r_Z(\mathcal{O}_X)\quad\text{and}$$
$$\mathrm{Gr}^F_{-n}\mathrm{DR}_X W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)=\omega_X\otimes_{\mathcal{O}_X}F_0W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X),$$

while the image of the inclusion $\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z,\omega_X)\hookrightarrow\omega_X\otimes_{\mathcal{O}_X}\mathcal{H}^r_Z(\mathcal{O}_X)$ is $\omega_X\otimes_{\mathcal{O}_X}E_0\mathcal{H}^r_Z(\mathcal{O}_X)$. We thus obtain the assertion in (b) in this case.

From now on we assume $k\geq 1$. Arguing by induction on k, we may and will assume that $F_pW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)=E_p\mathcal{H}^r_Z(\mathcal{O}_X)$ for $p\leq k-1$. In particular, we know that Z has $(k-1)$-Du Bois singularities, and thus $\mathrm{codim}_Z(Z_{\mathrm{sing}})\geq 2k-1\geq k$ by [MP22a, Corollary 3.40]. We only need to prove that the injection $F_kW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)\hookrightarrow E_k\mathcal{H}^r_Z(\mathcal{O}_X)$ is indeed an isomorphism. Moreover, because we know the corresponding assertions for the lower pieces of the filtrations, it follows from Lemma 2.1 that it is enough to show that the induced morphism

$$\mathrm{Gr}^F_{k-n}\mathrm{DR}_X W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)\to \mathrm{Gr}^E_{k-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X)\tag{24}$$

is an isomorphism (in the derived category).

Applying $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(-,\omega_X[r+k]\big)$ to the isomorphism in (e) implies via (1) that the composition

$$\mathrm{Gr}^F_{k-n}\mathrm{DR}_X W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X)\to \mathrm{Gr}^F_{k-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X)\to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\Omega^k_Z,\omega_X[r+k]\big)\tag{25}$$

is an isomorphism. On the other hand, since $\mathrm{codim}_Z(Z_{\mathrm{sing}})\geq k$, it follows from [MP22a, Section 5.2] that the second map in (25) gets identified with the canonical morphism

$$\mathrm{Gr}^F_{k-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X)\to \mathrm{Gr}^E_{k-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X).$$

We thus conclude that indeed (24) is an isomorphism.

Step 6. Proof of (b)⇒(a). In addition to the map $\sigma\colon \big(\mathrm{Gr}^{r-1}_V B_{\mathbf{f}}\big)^{\oplus r}\xrightarrow{(t_1,t_2,\ldots,t_r)}\mathrm{Gr}^r_V B_{\mathbf{f}}$ that we used in Step 1, we also consider the map $\delta\colon \mathrm{Gr}^r_V B_{\mathbf{f}}\xrightarrow{(\partial_{t_1},\partial_{t_2},\ldots,\partial_{t_r})}\big(\mathrm{Gr}^r_V B_{\mathbf{f}}(-1)\big)^{\oplus r}$. The key point is to show the following

Claim. The hypothesis in (b) implies that the composition of the canonical morphisms

$$\mathrm{Gr}^F_{k+r}\ker\delta\hookrightarrow \mathrm{Gr}^F_{k+r}\mathrm{Gr}^r_V B_{\mathbf{f}}\twoheadrightarrow \mathrm{Gr}^F_{k+r}\operatorname{coker}\sigma,\tag{26}$$

is an isomorphism.

By [CD21, Theorem 1.2], we have an isomorphism of filtered $\mathcal{D}_X$-modules

$$\mathrm{Gr}^W_{d+i}i_*\mathbf{Q}^H_Z[d]\cong \mathrm{Gr}^W_{n+i}\ker\delta\quad\text{for all}\quad i\in\mathbf{Z}.\tag{27}$$

In particular, we have $W_n\ker\delta=\ker\delta$ (recall that, similarly, the isomorphism (19) implies $W_{n-1}\operatorname{coker}\sigma=0$). Note that the inclusion $W_n\ker\delta\hookrightarrow W_n\mathrm{Gr}^r_V B_{\mathbf{f}}$ induces a canonical filtered morphism

$$\mathrm{Gr}^W_n\ker\delta\to W_n\operatorname{coker}\sigma=\frac{W_n\mathrm{Gr}^r_V B_{\mathbf{f}}}{W_n\mathrm{Gr}^r_V B_{\mathbf{f}}\cap\operatorname{im}\sigma}.\tag{28}$$

Indeed, the morphsim is well-defined because

$$W_{n-1}\ker\delta \subseteq W_{n-1}\mathrm{Gr}_V^r B_{\mathbf{f}} = (s+r)\cdot W_{n+1}\mathrm{Gr}_V^r B_{\mathbf{f}} = \sum_{i=1}^r t_i\partial_{t_i} W_{n+1}\mathrm{Gr}_V^r B_{\mathbf{f}} \subseteq \mathrm{im}\,\sigma.$$

We deduce that the canonical map $\ker\delta \to \mathrm{coker}\,\sigma$ factors as

$$\ker\delta \to \mathrm{Gr}_n^W \ker\delta \to W_n\,\mathrm{coker}\,\sigma \to \mathrm{coker}\,\sigma.$$

Furthermore, the canonical maps $\mathrm{Gr}_{k+r}^F \ker\delta \to \mathrm{Gr}_{k+r}^F\mathrm{Gr}_n^W \ker\delta$ and $\mathrm{Gr}_{k+r}^F W_n\,\mathrm{coker}\,\sigma \to \mathrm{Gr}_{k+r}^F\,\mathrm{coker}\,\sigma$ are isomorphisms because of (19) and (27), together with the fact that the canonical map

$$\mathrm{Gr}_{p+r}^F i_*\mathbf{Q}_Z^H[d] \to \mathrm{Gr}_{p+r}^F\mathrm{Gr}_d^W i_*\mathbf{Q}_Z^H[d]$$

is an isomorphism for $p \leq k+1$ by Lemma 3.5. Therefore the claim is now reduced to the assertion that (28) is a filtered isomorphism. Clearly, (28) is a filtered isomorphism over the regular locus Z_{reg} due to

$$\ker\delta|_{Z_{\mathrm{reg}}} = \mathrm{Gr}_V^r B_{\mathbf{f}}|_{Z_{\mathrm{reg}}} = \mathrm{coker}\,\sigma|_{Z_{\mathrm{reg}}} = \iota_+\mathcal{O}_{Z_{\mathrm{reg}}},$$

preserving the Hodge filtration, where $\iota\colon X \to X\times\mathbf{A}^r$ is the graph embedding corresponding to the functions $f_1, f_2, \ldots, f_r$. Therefore (28) is an isomorphism of $\mathcal{D}_X$-modules because its source and target decompose by (27) and (19) as direct sums of simple $\mathcal{D}$-modules, corresponding to the irreducible components of Z. Moreover, it is even a filtered isomorphism thanks to the fact that the Hodge filtration is uniquely determined by the regular locus [Sai88, (3.2.2.2)]. This completes the proof of the claim.

To conclude the proof, note that the assertion in (b) implies that $\widetilde{\alpha}(Z) \geq k+r$ by Theorem 2.3. Therefore we have

$$\mathrm{Gr}_{k+r}^F\mathrm{Gr}_V^r B_{\mathbf{f}} = \mathrm{Gr}_{k+r}^F B_{\mathbf{f}}/I_Z\cdot\mathrm{Gr}_{k+r}^F B_{\mathbf{f}} = \mathrm{Gr}_{k+r}^F\,\mathrm{coker}\,\sigma,$$

where the first equality follows from the fact that $F_{k+r}V^{>r}B_{\mathbf{f}} = \sum_{i=1}^r t_i\cdot F_{k+r}V^{>r-1}B_{\mathbf{f}}$ by [CD21, Theorem 1.1]. The claim implies that the composition

$$\mathrm{Gr}_{k+r}^F\ker\delta \hookrightarrow \mathrm{Gr}_{k+r}^F\mathrm{Gr}_V^r B_{\mathbf{f}} \xrightarrow{=} \mathrm{Gr}_{k+r}^F\,\mathrm{coker}\,\sigma,$$

is an isomorphism, hence δ is zero on $\mathrm{Gr}_{k+r}^F\mathrm{Gr}_V^r B_{\mathbf{f}} = \mathrm{Gr}_{k+r}^F B_{\mathbf{f}}/I_Z\cdot\mathrm{Gr}_{k+r}^F B_{\mathbf{f}}$. Therefore

$$\partial_{t_i}\cdot F_{k+r}B_{\mathbf{f}} \subseteq F_{k+r}B_{\mathbf{f}} + V^{>r-1}B_{\mathbf{f}} \subseteq V^r B_{\mathbf{f}} + V^{>r-1}B_{\mathbf{f}} \subseteq V^{>r-1}B_{\mathbf{f}} \quad \text{for} \quad 1 \leq i \leq r,$$

where the second inclusion comes from the fact that $\widetilde{\alpha}(Z) \geq k+r$. This implies $F_{k+r+1}B_{\mathbf{f}} \subseteq V^{>r-1}B_{\mathbf{f}}$, which is equivalent to $\widetilde{\alpha}(Z) > k+r$. This completes the proof of this step and thus the proof of the theorem. ☐

We next prove the two corollaries stated in the Introduction:

Proof of Corollary 1.2. The assertion follows from the fact that Z has k-Du Bois singularities if and only if $\widetilde{\alpha}(Z) \geq k+r$, while by Theorem 1.1, Z has $(k-1)$-rational singularities if and only if $\widetilde{\alpha}(Z) > k+r-1$. ☐

Proof of Corollary 1.3. We may assume that Z is irreducible and affine and let $Z \hookrightarrow X$ be a closed embedding, of codimension r, with X a smooth, irreducible variety. The assertion to prove is trivial if Z is smooth (with the convention that the empty set has infinite codimension), hence we may and will assume that Z is singular. If $s = \dim(Z_{\mathrm{sing}})$ and H is the intersection of general hyperplanes sections in X, then $Z' := Z\cap H$ is a locally complete intersection variety with nonempty, 0-dimensional singular locus, and $\widetilde{\alpha}(Z') = \widetilde{\alpha}(Z)$ by

[CDMO22, Theorem 1.2]. In particular, it follows from Theorem 1.1 that $\widetilde{\alpha}(Z') > k+r$. Since $\mathrm{codim}_Z(Z_{\mathrm{sing}}) = \mathrm{codim}_{Z'}(Z'_{\mathrm{sing}})$, we may replace Z by Z' to assume that Z_{sing} is nonempty and zero-dimensional. We then need to show that $d := \dim(Z) \geq 2k+2$.

Let $x \in Z_{\mathrm{sing}}$. By (8), we have

$$\widetilde{\alpha}(Z) \leq \dim(X) - \frac{1}{2}\dim_{\mathbf{C}} T_x(Z) = (d+r) - \frac{1}{2}\dim_{\mathbf{C}} T_x(Z).$$

Since $x \in Z_{\mathrm{sing}}$, we have $\dim_{\mathbf{C}} T_x(Z) \geq \dim(Z) + 1 = d+1$, hence

$$\widetilde{\alpha}(Z) \leq (d+r) - \frac{1}{2}(d+1) = \frac{d-1}{2} + r.$$

Since $\widetilde{\alpha}(Z) > k+r$, we conclude that $k + r < \frac{d-1}{2} + r$, hence $d > 2k+1$. We thus conclude that $d \geq 2k+2$. $\square$

4. Non-vanishing result for the Du Bois complex

In this section we show that for singular, d-dimensional, locally complete intersection varieties Z with k-rational singularities, where $k \geq 1$, the cohomology sheaf $\mathcal{H}^k(\underline{\Omega}_Z^{d-k})$ does not vanish.

Proof of Theorem 1.5. Note first that Theorem 2.5 gives an isomorphism

$$\Omega_Z^k \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(\underline{\Omega}_Z^{d-k}, \omega_Z),$$

and since $\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(-, \omega_Z)$ is a duality, we get an isomorphism

$$\underline{\Omega}_Z^{d-k} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(\Omega_Z^k, \omega_Z).$$

The first isomorphism in the theorem follows by taking that k-th cohomology sheaf.

It is shown in [MP22a, Section 5.2] that since Z has k-Du Bois singularities (more precisely, since $\mathrm{codim}_Z(Z_{\mathrm{sing}}) \geq k$), the sheaf Ω_Z^k is the 0-th cohomology of the complex

$$0 \to \mathrm{Sym}^k(\mathcal{N}_{Z/X})^\vee \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathrm{Sym}^{k-1}(\mathcal{N}_{Z/X})^\vee \to \ldots \to \Omega_X^{k-1} \otimes_{\mathcal{O}_X} \mathcal{N}_{Z/X}^\vee \to \Omega_X^k \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to 0,$$

placed in cohomological degrees $-k, \ldots, 0$. Since this is a resolution of Ω_Z^k by locally free $\mathcal{O}_Z$-modules, it follows that

$$\mathcal{E}xt^k_{\mathcal{O}_Z}(\Omega_Z^k, \omega_Z) \simeq \omega_Z \otimes_{\mathcal{O}_Z} \mathcal{E}xt^k_{\mathcal{O}_Z}(\Omega_Z^k, \mathcal{O}_Z)$$

$$\simeq \omega_Z \otimes_{\mathcal{O}_Z} \mathrm{coker}\big(\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathrm{Sym}^{k-1}_{\mathcal{O}_Z}(\mathcal{N}_{Z/X}) \to \mathrm{Sym}^k_{\mathcal{O}_Z}(\mathcal{N}_{Z/X})\big) \simeq \omega_Z \otimes_{\mathcal{O}_Z} \mathrm{Sym}^k_{\mathcal{O}_Z}(\mathcal{Q}),$$

where the last isomorphism follows from [Eis95, Proposition A2.2(d)].

In order to see that $\mathcal{H}^k(\underline{\Omega}_Z^{d-k})_x \neq 0$ if $x \in Z$ is a singular point, it is enough to consider, in a neighborhood of x, a closed immersion $Z \hookrightarrow X$ such that $T_xZ = T_xX$. In this case the morphism of locally free $\mathcal{O}_Z$-modules

$$\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathrm{Sym}^{k-1}_{\mathcal{O}_Z}(\mathcal{N}_{Z/X}) \to \mathrm{Sym}^k_{\mathcal{O}_Z}(\mathcal{N}_{Z/X})$$

is given by a matrix whose entries all vanish at x. We thus conclude that the minimal number of generators of $\mathcal{H}^k(\underline{\Omega}_Z^{d-k})_x$ is equal to $\mathrm{rank}\big(\mathrm{Sym}^k_{\mathcal{O}_Z}(\mathcal{N}_{Z/X})\big) = \binom{e-d+k-1}{k}$, where $e = \dim_{\mathbf{C}} T_xZ$, hence it is nonzero since $e \geq d+1$. This concludes the proof. $\square$

5. Generation level of the Hodge filtration in terms of the minimal exponent

In this section we prove the bound on the level of generation in terms of the minimal exponent.

Proof of Theorem 1.7. Let $d=\dim(Z)=n-r$. The starting point is the observation that for every mixed Hodge module M on X, the Hodge filtration is generated at level q if $\mathcal{H}^0{\rm Gr}^F_{p-n}{\rm DR}_X(M)=0$ for all $p>q$. Recall that by (1), we have

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(M)\simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{n-p}\mathrm{DR}_X(\mathbf{D}(M)),\omega_X[n]\big).$$

If we apply this with $M=\mathcal{H}^ri_*i^!\mathbf{Q}_X^H[n]=\mathcal{H}^r_Z(\mathcal{O}_X)$, where $i\colon Z\hookrightarrow X$ is the inclusion, since $\mathbf{D}(M)\simeq i_*\mathbf{Q}_Z^H[d](n)$, we conclude that

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_X\mathcal{H}^r_Z(\mathcal{O}_X)\simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{-p}\mathrm{DR}_Xi_*\mathbf{Q}_Z^H[d],\omega_X[n]\big).$$

If we apply $\mathcal{H}^0(-)$ on both sides, we conclude that the Hodge filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$ is generated at level q if

$$\mathcal{E}xt^n_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{-p}\mathrm{DR}_Xi_*\mathbf{Q}_Z^H[d],\mathcal{O}_X\big)=0 \tag{29}$$

for all $p>q$.

Recall now that for a bounded complex of $\mathcal{O}_X$-modules $K^\bullet$ and an $\mathcal{O}_X$-module $\mathcal{F}$, there is a spectral sequence

$$E_1^{i,j}=\mathcal{E}xt^j_{\mathcal{O}_X}(K^{-i},\mathcal{F})\implies \mathcal{E}xt^{i+j}_{\mathcal{O}_X}(K^\bullet,\mathcal{F}).$$

We take $K^\bullet$ to be the complex $\mathrm{Gr}^F_{-p}\mathrm{DR}_Xi_*\mathbf{Q}_Z^H[d]$, so that

$$K^{-\ell}=\Omega_X^{n-\ell}\otimes_{\mathcal{O}_X}\mathrm{Gr}^F_{n-\ell-p}i_*\mathbf{Q}_Z^H[d].$$

Therefore the vanishing in (29) holds if for all $j\in\{0,\ldots,n\}$, we have

$$\mathcal{E}xt^j_{\mathcal{O}_X}\big(\Omega_X^{n-(n-j)}\otimes_{\mathcal{O}_X}\mathrm{Gr}^F_{n-(n-j)-p}i_*\mathbf{Q}_Z^H[d],\omega_X\big)=0,$$

or equivalently,

$$\mathcal{E}xt^j_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{j-p}i_*\mathbf{Q}_Z^H[d],\mathcal{O}_X\big)=0.$$

We conclude that in order to complete the proof of the theorem, it is enough to show the following claim:

Claim 5.1. For all $p\geq n-\lceil\widetilde{\alpha}(Z)\rceil$ and all $j\in\{0,1,\ldots,n\}$, we have

$$\mathcal{E}xt^j_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{j-p}i_*\mathbf{Q}_Z^H[d],\mathcal{O}_X\big)=0. \tag{30}$$

In order to prove the claim, we may and will assume that X is affine and Z is defined by $f_1,\ldots,f_r\in\mathcal{O}_X(X)$, so we can make use of the corresponding V-filtration. By Theorem 2.2, for every $\ell\in\mathbf{Z}$, we have an isomorphism

$$F_\ell i_*\mathbf{Q}_Z^H[d]\simeq\ker\big(F_\ell\mathrm{Gr}_V^r(B_\mathbf{f})\overset{\partial_{t_1},\ldots,\partial_{t_r}}{\longrightarrow}\bigoplus_{i=1}^rF_\ell\mathrm{Gr}_V^{r-1}(B_\mathbf{f})(-1)\big).$$

Suppose now that $\widetilde{\alpha}(Z)>\ell$. In this case, by definition of the minimal exponent we have $F_{\ell+1}B_\mathbf{f}\subseteq V^{>r-1}B_\mathbf{f}$ and $F_\ell B_\mathbf{f}\subseteq V^rB_\mathbf{f}$. We thus conclude that

$$F_\ell i_*\mathbf{Q}_Z^H[d]\simeq F_\ell\mathrm{Gr}_V^r(B_\mathbf{f})\simeq F_\ell B_\mathbf{f}/F_\ell V^{>r}B_\mathbf{f}. \tag{31}$$

On the other hand, it follows from [CD21, Theorem 1.1] that we have

$$F_\ell V^{>r} B_{\mathbf{f}} = \sum_{i=1}^r t_i \cdot F_\ell V^{>r-1} B_{\mathbf{f}} = \sum_{i=1}^r t_i \cdot F_\ell B_{\mathbf{f}},$$

so that (31) gives

$$F_\ell i_* \mathbf{Q}_Z^H[d] \simeq F_\ell B_{\mathbf{f}} / (t_1, \ldots, t_r) F_\ell B_{\mathbf{f}},$$

and thus

$$\mathrm{Gr}_\ell^F i_* \mathbf{Q}_Z^H[d] \simeq \mathrm{Gr}_\ell^F B_{\mathbf{f}} / (t_1, \ldots, t_r) \mathrm{Gr}_\ell^F B_{\mathbf{f}}.$$

Recall now that by definition we have $F_\ell B_{\mathbf{f}} = \mathrm{Gr}_\ell^F B_{\mathbf{f}} = 0$ if $\ell < r$ and $\mathrm{gr}_\ell^F B_{\mathbf{f}} = \bigoplus_{|\beta|=\ell-r} \mathcal{O}_X \partial_t^\beta$ if $\ell \geq r$, with each t_i acting as multiplication by f_i. We thus conclude that if $\ell \geq r$, then

$$\mathrm{Gr}_\ell^F i_* \mathbf{Q}_Z^H[d] \simeq \bigoplus_{|\beta|=\ell-r} \mathcal{O}_Z \partial_t^\beta. \tag{32}$$

We now proceed to prove the claim. Note that since Z is singular, it follows from (8) that $\widetilde{\alpha}(Z) \leq n - \frac{1}{2}(d+1)$, hence $n - \lceil \widetilde{\alpha}(Z) \rceil \geq \lfloor (d+1)/2) \rfloor \geq 1$.

We first consider the case when $p > n - \lceil \widetilde{\alpha}(Z) \rceil$, so that $p > 0$ and $\lceil \widetilde{\alpha}(Z) \rceil > n - p \geq j - p$ for all $j \in \{0, 1, \ldots, n\}$. By taking $\ell = j - p$, it follows from (32) that we have

$$\mathrm{Gr}_{j-p}^F i_* \mathbf{Q}_Z^H[d] \simeq \begin{cases} 0 & \text{if } j - p < r \\ \bigoplus_{|\beta|=j-p-r} \mathcal{O}_Z \partial_t^\beta & \text{if } j - p \geq r \end{cases}.$$

Clearly, the vanishing in (30) holds if $j - p < r$. If $j \geq r + p > r$, we use the fact that Z is a complete intersection, so we have locally the Koszul resolution of $\mathcal{O}_Z$, of length r, by free $\mathcal{O}_X$-modules. In particular, we have $\mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) = 0$ for all $j > r$, proving the claim in this case.

We next consider the case when $p = n - \lceil \widetilde{\alpha}(Z) \rceil$. If $j \in \{0, 1, \ldots, n-1\}$, then $\lceil \widetilde{\alpha}(Z) \rceil > j - p$ and we get the vanishing in (30) as above. In order to complete the proof of the claim, it is thus enough to consider $j = n$ and show that

$$\mathcal{E}xt^n_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} i_* \mathbf{Q}_Z^H[d], \mathcal{O}_X\big) = 0. \tag{33}$$

It follows from Theorem 2.2 that we have an inclusion $\mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} i_* \mathbf{Q}_Z^H[d] \subseteq \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} \mathrm{Gr}_V^r B_{\mathbf{f}}$. Since $\mathcal{E}xt^{n+1}_{\mathcal{O}_X}(-, \mathcal{O}_X) = 0$, we deduce using the long exact sequence of $\mathcal{E}xt$ sheaves that we have a surjection

$$\mathcal{E}xt^n_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} \mathrm{Gr}_V^r B_{\mathbf{f}}, \mathcal{O}_X\big) \to \mathcal{E}xt^n_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} i_* \mathbf{Q}_Z^H[d], \mathcal{O}_X\big).$$

Therefore it is enough to show that the left term is 0.

Note now that it follows from [CD21, Theorem 1.1] that

$$F_{\lceil \widetilde{\alpha}(Z) \rceil} V^{>r} B_{\mathbf{f}} = (t_1, \ldots, t_r) F_{\lceil \widetilde{\alpha}(Z) \rceil} V^{>r-1} B_{\mathbf{f}} = (t_1, \ldots, t_r) F_{\lceil \widetilde{\alpha}(Z) \rceil} B_{\mathbf{f}},$$

where the second equality follows from the definition of the minimal exponent. Therefore we have

$$\begin{aligned} \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} \mathrm{Gr}_V^r B_{\mathbf{f}} &= \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} V^r B_{\mathbf{f}} / (t_1, \ldots, t_r) \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} B_{\mathbf{f}} \\ &\subseteq \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} B_{\mathbf{f}} / (t_1, \ldots, t_r) \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} B_{\mathbf{f}}. \end{aligned}$$

Using again the fact that $\mathcal{E}xt^{n+1}_{\mathcal{O}_X}(-, \mathcal{O}_X) = 0$, we see that it is enough to show that

$$\mathcal{E}xt^n_{\mathcal{O}_X}(\mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} B_{\mathbf{f}} / (t_1, \ldots, t_r) \mathrm{Gr}^F_{\lceil \widetilde{\alpha}(Z) \rceil} B_{\mathbf{f}}, \mathcal{O}_X) = 0.$$

This follows from the fact that $\mathrm{Gr}^F_{\lceil\widetilde{\alpha}(Z)\rceil}B_{\mathbf{f}}/(t_1,\ldots,t_r)\mathrm{Gr}^F_{\lceil\widetilde{\alpha}(Z)\rceil}B_{\mathbf{f}}$ is isomorphic to a direct sum of copies of $\mathcal{O}_Z$ and $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_Z,\mathcal{O}_X)=0$, as follows using the Koszul resolution of $\mathcal{O}_Z$ (note that $r<n$, since we assume that Z is reduced and singular). This completes the proof of the claim and thus the proof of the theorem. □

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